Math work Book
June 7, 2018

Authors
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Student’s Name: ___________________________  Student ID: ________________

Instructor: ________________________________

• Show work for full credit.
• Points will be given to the correct steps leading to the correct answer.
• All work should be done on the EXAM itself.
• Calculator use is allowed.

Wish you a happy semester!

For use by instructor only.

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**Preface** Mat 106 is a general education course which is a core requirement for all non science majors. Roughly, 250 students enrolled in Mat 106 every year. Although, success rate is satisfactory, department attempts to provide a highest quality to these students. This workbook is designed to address two subjects: (1) To help student become independent learner and (2) To provide a clear guideline on how the course is developed. More blah blah blah.
1 Basic Algebra

1.1 Order of operations - PEMDAS

1.1.1 Tutorial

Simplifying a complex expression involving numbers is generally done using PEMDAS, where

- we first attempt to simplify P, that is a part or parts of expression which are in parenthesis. Sometimes, we might have to use entire PEMDAS to simplify one parenthesis!

- E (exponent), M (multiplication), D (division), A (addition) and S (subtraction) are then used in the same order. We repeat this process until we reach to a single number (or a simple expression).

- we will use scientific calculator when needed.

Hot tips

- Sometimes we use invisible parenthesis, for example numerator of the fraction $\frac{2 \times 7 - 13}{4}$ is in a parenthesis. We should actually write $\frac{(2 \times 7 - 13)}{4}$.

- When a whole number or an expression without a numerator is added, subtracted or multiplied to a fraction, we use 1 as a denominator. For example,

$$23 + \frac{17}{49} = \frac{23}{1} + \frac{17}{49} \text{ or } 6 \times \frac{7}{4} = \frac{6}{1} \times \frac{7}{4}$$

- Following expressions are all different!

$$\frac{2 \times 7 - 13}{4}, \quad 2 \times \frac{7 - 13}{4} \quad \text{and} \quad 2 \times 7 - \frac{13}{4},$$

Number story Observe the following pattern:

<table>
<thead>
<tr>
<th>First Column</th>
<th>Second Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 8 + 1 = 9$</td>
<td>$0 \times 9 + 1 = 1$</td>
</tr>
<tr>
<td>$12 \times 8 + 2 = 98$</td>
<td>$1 \times 9 + 2 = 11$</td>
</tr>
<tr>
<td>$123 \times 8 + 3 = 987$</td>
<td>$12 \times 9 + 3 = 111$</td>
</tr>
<tr>
<td>$1234 \times 8 + 4 = 9876$</td>
<td>$123 \times 9 + 4 = 1111$</td>
</tr>
</tbody>
</table>

Guess the next row and check your answer!
1.1.2 Examples

1. Simplify: \(3(5 - 3)^2 - 7(3 - 2 \times 4) + 10\).

→ **Solution:** This expression has two parenthesis, \((5 - 3)\) and \((3 - 2 \times 4)\). We first simplify them. Clearly \((5 - 3) = 2\) and \((3 - 2 \times 4) = (3 - 8) = -5\). Therefore,

\[
[1] 3(5 - 3)^2 - 7(3 - 2 \times 4) + 10 = 3(2)^2 - 7(3 - 8) + 10 \quad (P \text{ and } M)
\]

\[
= 3(4) - 7(-5) + 10 \quad (E)
\]

\[
= 12 + 35 + 10 \quad (M)
\]

\[
= 57 \quad \text{← answer}
\]

2. Simplify: \(2(6 - 17 + 2) - 3(-16 + 3^2) - 4(5 - 3\frac{1}{2})\).

→ **Solution:** We will first replace \(3\frac{1}{2}\) by 3.5 and then simplify parenthesis:

\[
2(6 - 17 + 2) - 3(-16 + 9) - 4(1.5) = 2(-9) - 3(-7) - 6 = -18 + 21 - 7 = -25 + 21 = -4
\]

3. Simplify: \(12 \times \frac{3(8 - 2) - 6}{4} + 5(4 + 2 \times 3 - 17) - 6\).

→ **Solution:** This expression also has two parenthesis, \((3(8 - 2) - 6)\) and \((4 + 2 \times 3 - 17)\) We first simplify them.

\[
\frac{3(8 - 2) - 6}{4} = \frac{3(6) - 6}{4} = \frac{18 - 6}{4} = \frac{12}{4} = 3
\]

\[
(4 + 2 \times 3 - 17) = (4 + 6 - 17) = -7
\]

Combining these calculations with the original questions, we get

\[
12 \times \frac{3(8 - 2) - 6}{4} + 5(4 + 2 \times 3 - 17) - 6 = 12 \times \frac{3(6) - 6}{4} + 5(4 + 6 - 17) - 6 \quad (P \text{ and } M)
\]

\[
= 12 \times \frac{18 - 6}{4} + 5(-10 - 17) - 6 \quad (E) = 12 \times \frac{12}{4} + 5(-7) - 6 \quad (M) = (12)(3) - 35 - 6 = 36 - 35 - 6
\]

\[
= 36 - 41 = -7 \quad \text{← answer}
\]

**Note:** In both examples, letters E, M etc are used to show steps. In practice, we don’t write them.

4. Find the value of \(40 \left(1 + \frac{0.03}{4}\right)^8\).

→ **Solution:** We us a scientific calculator to find the final answer. With most calculators, if you place parenthesis in right place (or places), then the answer will be correct. Since you also have to show work, we simplify this expression in several steps.

\[
40 \left(1 + \frac{0.03}{4}\right)^8 = 40(1 + 0.0075)^8 = 40(1.0075)^8
\]

\[
= 40(1.0615988) \text{ rounded to } 7 \text{ decimals}
\]

\[
= 42.463952 \quad \text{← ans.}
\]

5. If \(x = 3, y = 4\), find \(x^3 - y^2\).

→ **Solution:**

\[
x^3 - y^2 = (3)^3 - (4)^2 = 27 - 16 = 11 \quad \text{← ans.}
\]
1. Simplify: $3^2 - (5 + 2)^2 + 17$.

2. Simplify: $3(12 - 4 \times 5) + 4(6 - 15 + 2^2) - 2(8 - 22 + 11)$.

3. Simplify: $\frac{2(3+5\times2-7)}{2} + 1.5$.

4. Find the value of $x^3 - 3y^2 + 2xy + 11$ when $x = 2$ and $y = 3$.

5. Simplify: $25 \left(1 + \frac{0.06}{12}\right)^{12}$.
1.2 Linear Equations

1.2.1 Tutorial

Expression $2(3x - 5) = 5(x + 4)$ is an example of a linear equation, where $x$ is the variable. Solution (or an answer) always looks like $x = \text{some number}$. Process to solve a linear equation is to bring variables on one side and numbers on the other side of equality sign ‘$=$’ using various algebraic operations such as addition, subtractions, multiplication.

1. **Standard form**: $ax + b = c$ 
   **Solution**: $x = \frac{c-b}{a}$, $a \neq 0$.

2. **Distributive property** $a(x + b) = ax + ab$ or for more terms $a(b + c + d + \ldots) = ab + ac + ad + \ldots$

3. If we add, subtract, multiply or divide both sides of an equation by same nonzero number, then solution of the resulting equation is unchanged.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Step</th>
<th>Result</th>
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</thead>
<tbody>
<tr>
<td>$x - 7 = 10$</td>
<td>Add 7 on both sides</td>
<td>$x = 17$</td>
</tr>
<tr>
<td>$x + 12 = 3$</td>
<td>Subtract 12 from both sides</td>
<td>$x = -9$</td>
</tr>
<tr>
<td>$\frac{x}{5} = 20$</td>
<td>Multiply by 5 to both sides</td>
<td>$x = 100$</td>
</tr>
<tr>
<td>$4x = 13$</td>
<td>Divide both sides by 4</td>
<td>$x = 3.25$</td>
</tr>
</tbody>
</table>

Table 1: Four steps

Linear equation story: Method we have used here is called the elimination method. By employing basic operations, we eliminate terms in such a way that we end up with “variable=number” form which is the answer of the equation. What if we end up with a weird expression?

<table>
<thead>
<tr>
<th>Result</th>
<th>Example</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable = zero</td>
<td>$7x=0$</td>
<td>$x=0$</td>
</tr>
<tr>
<td>zero = a number</td>
<td>$0 = 5$</td>
<td>No solution</td>
</tr>
</tbody>
</table>
1.2.2 Examples

1. Solve for $x$: $7x + 1 = 22$.

→ Solution:

$$7x + 1 = 22$$
$$-1 = -1$$
$$\Rightarrow 7x = 21$$
$$\Rightarrow \frac{7x}{7} = \frac{21}{7} \text{ or } x = 3 \leftarrow \text{answer}$$

2. Solve for $t$: $3t - 5 = 2(t + 1)$.

→ Solution:

$$3t - 5 = 2(t + 1)$$
$$\Rightarrow 3t - 5 = 2t + 2$$
$$+5 = +5$$
$$3t = 2t + 7$$
$$-2t = -2t \text{ or } t = 7 \leftarrow \text{answer}$$

3. Solve for $x$: \(\frac{2x + 5}{3} = \frac{x + 7}{4}\).

→ Solution:

$$\frac{2x + 5}{3} = \frac{x + 7}{4} \text{ multiply both sides by } \frac{(3)(4)}{1}$$
$$\left(\frac{3}{4}\right)(4) \times \frac{2x + 5}{3} = \left(\frac{3}{4}\right)(4) \times \frac{x + 7}{4}$$
$$\Rightarrow 4(2x + 5) = 3(x + 7)$$
$$\Rightarrow 8x + 20 = 3x + 21 \text{ distributive property}$$
$$-20 = -20$$
$$\Rightarrow 8x = 3x + 1$$
$$-3x = -3x$$
$$\Rightarrow 5x = 1 \text{ or } x = 0.2 \leftarrow \text{answer}$$

4. Solve for $u$: $3(u - 2) + u = 2(2u + 1)$.

→ Solution:

$$3(u - 2) + u = 2(2u + 1)$$
$$3u - 6 + u = 4u + 2$$
$$4u - 6 = 4u + 2$$
$$4u = 4u + 8$$
$$0 = 8$$

This equation has no solution. It is inconsistent.
1.2.3 Homework

1. Solve for $x$: $2x - 5 = 7$.

2. Solve for $t$: $5t - 9 = 3(t + 1)$.

3. Solve for $u$: $2(3u - 5) + 3(2u + 2) = 4(3u - 2)$.

4. Solve for $w$: $\frac{2w - 1}{3} = \frac{3w + 2}{4}$.

5. Solve for $x$: $\frac{x + 3}{2x - 1} = \frac{2}{3}$. 
1.3 Formulas

1.3.1 Tutorial

Formulas provide a useful computational tool used in many real life problems. Formulas are equations which involve two or more variables. In this section we want to learn (1) how to substitute values in a formula and (2) how to change a subject of a formula.

<table>
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<th>Subject</th>
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<td>Triangle</td>
<td>Area $A = \frac{1}{2}bh$</td>
</tr>
<tr>
<td></td>
<td>where $b$=Base and $h$=height.</td>
</tr>
<tr>
<td></td>
<td>Perimeter = sum of the three sides.</td>
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<tr>
<td>Rectangle</td>
<td>Area $= L \times W$</td>
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<td></td>
<td>Perimeter $P = 2(L + W)$</td>
</tr>
<tr>
<td></td>
<td>where $L$=length and $W$=width.</td>
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<tr>
<td>Square</td>
<td>Area $= L^2$</td>
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<tr>
<td></td>
<td>Perimeter $P = 4L$</td>
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<tr>
<td></td>
<td>where $L$=length.</td>
</tr>
<tr>
<td>Circle</td>
<td>Area $A = \pi r^2$</td>
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<tr>
<td></td>
<td>circumference $C = 2\pi r$</td>
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<tr>
<td></td>
<td>where $r$=radius.</td>
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<td>Simple</td>
<td>Interest $I = PrT$</td>
</tr>
<tr>
<td>interest</td>
<td>Final balance $A = P + I$</td>
</tr>
<tr>
<td></td>
<td>where $P$ = initial investment</td>
</tr>
<tr>
<td></td>
<td>$r$ =annual rate and $T$ = time in years.</td>
</tr>
<tr>
<td>Temperature</td>
<td>Conversion</td>
</tr>
<tr>
<td></td>
<td>$F = 1.8C + 32$ and $C = \frac{F-32}{1.8}$</td>
</tr>
<tr>
<td></td>
<td>where $F$ = Fahrenheit and $C$ = Celsius.</td>
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Table 2: Some important formulas

**Counting story:** A young man approached a rich businessman with an offer. “I have” he said, “32 gold coins, I want to exchange them for some rice grains. For the first coin, I want one grain. For the second coin, I want double, which is two grains. Continuing this way, I want twice as much rice grains as I got for the previous coin.” “Idiot” Businessman thought and accepted the offer. Here is the grain calculation:

<table>
<thead>
<tr>
<th>Days</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grains</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
<td>512</td>
<td>1024</td>
<td>2048</td>
<td>4096</td>
<td>8192</td>
<td>16,384</td>
</tr>
</tbody>
</table>

For the $25^{th}$ coin, he will get 16,777,216 number of grains. For the $32^{nd}$ coin, he will get 4,294,967,296 number of grains (which is roughly 200,000 pounds). All together (for 32 coins) he got 8,589,934,591 number of grains! Present value of 32 gold coins, each weighing 5 grams is roughly $64,000 and value of 800 billion rice grains is approximately 3 million dollars. There are formulas which can do these calculations.
1.3.2 Examples

1. Suppose perimeter $P$ of a rectangle is $P = 2L + 2W$ where $L$ is the length and $W$ is the width.

(a) Find the perimeter of a rectangle 10 m long and 8 m wide.
(b) Perimeter of a rectangle is 100 ft. Find the length if the width is 20 ft.
(c) Solve the formula for $L$ and find $L$ when $P = 100$ ft. and $W = 20$ ft.

→ Solution:

(a) Perimeter $P = 2L + 2W$ where $L = 10$ m and $W = 8$ m. Substituting these values in the formula gives, $P = 2(10) + 2(8) = 20 + 16 = 36$ m.
(b) Here, $P = 100$ ft and $W = 20$ ft. Substituting these values in $P = 2L + 2W$, we get $100 = 2L + 2(20) \implies 100 = 2L + 40$. Subtracting 40 from each sides, we get $100 = 2L$.
Dividing both sides by 2, we derive $L = 30$ ft.
(c) We use elimination method to solve $P = 2L + 2W$ From $P = 2L + 2W$ we subtract $2W$ from the either sides to get $P - 2W = 2L$.
Now divide both sides by 2 to get $L = \frac{P - 2W}{2}$.
To find $L$, we substitute $P = 100$ and $W = 20$.

$$L = \frac{P - 2W}{2} = \frac{100 - 2(20)}{2} = \frac{100 - 40}{2} = \frac{60}{2} = 30 \text{ feet}$$

2. Formula to compute body mass index is $BMI = \frac{W \times 0.45}{(H \times 0.025)^2}$, where $W$ is a weight in pound (lb.) and $H$ is a height in inches of a person. Find BMI of a person weighing 170 lb. who with 68 inches of height. If a healthy BMI is between 19 and 25, what would you advice him?

→ Solution: In the formula, we substitute $W = 170$ and $H = 68$ to get

$$BMI = \frac{170 \times 0.45}{(68 \times 0.025)^2} = \frac{76.5}{(1.07)^2} = \frac{76.5}{2.89} = 26.45$$

He needs to watch his weight.

3. Simple interest $I = Prt$ where $P$ is the initial investment, $r$ is the annual rate (in decimal) and $t$ is time.

(a) Find the simple interest on an investment of $2000 for 10 years with 4% annual rate of interest.
(b) What should be $P$ to get interest of $800 in 4 years if the rate of interest is 6%?.
(c) Solve the formula for $r$. What must be the annual rate so that the investment of $2000 earns $1,500 interest in 8 years?

→ Solution:

(a) Simple interest $I = Prt$ where the investment $P = 2000$, $t = 10$ years and $r = 4\% = 0.04$.
Substituting these values in the formula we get, $I = ($2000)(10)(0.04) = $800.
(b) We are given $I = 800$, $t = 4$ years and $r = 0.06$. Substituting these values in $I = Prt$, we get $800 = P(0.06)(4) \implies 800 = p(0.24)$. Dividing both sides by 0.24, we get $P = \frac{800}{0.24} = 3,333.33$.
(c) Simple interest formula is $I = Prt$. Dividing both sides by $Pt$, we get a new formula $r = \frac{I}{Pt}$.
Now $P = 2000$, interest amount $I = 1,500$ and $t = 8$. We substitute these values in our new formula:

$$r = \frac{I}{Pt} = \frac{1,500}{(2000)(8)} = \frac{1,500}{16,000} = 0.09375 \rightarrow r = 9.4\% \text{ ans.}$$
4. If the radius of a circle is \( r \) then the area \( a = \pi r^2 \) and circumference \( C = 2\pi r \).

(a) Find the area and circumference of a circle with radius \( r = 3 \text{ ft} \).

(b) Solve the formula for the circumference of a circle for \( r \) and find the radius of a circle with 28 feet of a boundary.

→ **Solution:**

(a) We use the formulas \( A = \pi r^2 \) and \( C = 2\pi r \) to find the area (A) and the circumference (C), where \( \pi = 3.1415 \) and \( r = 3 \).

\[
A = \pi r^2 = (3.1415)(3^2) = 28.2735 \text{ sq. ft} \quad \text{and} \quad C = 2\pi r = 2(3.1415)(3) = 18.8490 \text{ ft}
\]

(b) Circumference \( C = 2\pi r \). We divide both sides by \( 2\pi \) and rewrite the new formula as \( r = \frac{C}{2\pi} \). Now we substitute \( C = 28 \text{ ft} \) to get

\[
r = \frac{C}{2\pi} = \frac{28}{2(3.1415)} = \frac{28}{6.2830} = 4.4565
\]
1.3.3  **Homework**

1. Suppose perimeter $P$ of a rectangle is $P = 2L + 2W$ where $L$ is the length and $W$ is the width.
   
   (a) Find the perimeter of a rectangle 12 ft long and 6 ft wide.

   (b) Perimeter of a rectangle is 140 ft. Find the length if the width is 30 ft. What is the area of the rectangle?

2. Simple interest $I = Prt$ where $P$ is the initial investment, $r$ is the annual rate (in decimal) and $t$ is time.
   
   (a) Find the simple interest on an investment of $3500 for 7 years with 3.5% annual rate of interest.

   (b) What should be $P$ to get interest of $1200 in 5 years if the rate of interest is 4%?.

3. Temperature Conversion formula is $F = 1.8C + 32$ and $C = \frac{F-32}{1.8}$ where, $F$ stands for Fahrenheit and $C$ for Celsius.
   
   (a) Convert $90^\circ$ Fahrenheit to Celsius.

   (b) Convert $40^\circ$ Celsius to Fahrenheit.
1.4 Applications of linear equations

1.4.1 Tutorial

We translate English sentences into mathematical expression to solve word problems. Here are some examples:

<table>
<thead>
<tr>
<th>English Phrase</th>
<th>Mathematical expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Six more than a number</td>
<td>x + 6</td>
</tr>
<tr>
<td>A number increased by 3</td>
<td>x + 3</td>
</tr>
<tr>
<td>Four less than a number</td>
<td>x - 4</td>
</tr>
<tr>
<td>A number decreased by 9</td>
<td>x - 9</td>
</tr>
<tr>
<td>Twice a number</td>
<td>2x</td>
</tr>
<tr>
<td>Four times a number</td>
<td>4x</td>
</tr>
<tr>
<td>3 decreased by a number</td>
<td>3-x</td>
</tr>
<tr>
<td>The difference between a number and 5</td>
<td>x-5</td>
</tr>
<tr>
<td>Four less than 3 times a number</td>
<td>3x - 4</td>
</tr>
<tr>
<td>Ten more than twice a number</td>
<td>2x + 10</td>
</tr>
<tr>
<td>The sum of 5 times a number and 3</td>
<td>5x + 3</td>
</tr>
<tr>
<td>Eight times a number, decreased by 7</td>
<td>8x - 7</td>
</tr>
<tr>
<td>Six more than a number is 10.</td>
<td>x + 6 = 10</td>
</tr>
<tr>
<td>Five less than a number is 20.</td>
<td>x-5=20</td>
</tr>
<tr>
<td>Twice a number, decreased by 6 is 12.</td>
<td>2x - 6 = 1</td>
</tr>
<tr>
<td>A number decreased by 13 is 6 times the number.</td>
<td>x-13=6x</td>
</tr>
</tbody>
</table>

A toy story: In a toyshop, price of a dog is 50 cents, price of a horse is $1.50 and price of an elephant is $2.50. Bruce bought some dogs, some horses and some elephants (at least one of each). Altogether, he bought 100 toys and paid $100 dollars. How many dogs, how many horses and how many elephants did he buy?
1.4.2 Examples

1. A number increased by 9 is 5 less than twice the number. Find the number.

→ Solution: Suppose the number is \( n \). Let us rewrite the statement in mathematical language:

\[
\text{A number increased by 9 is 5 less than twice the number} \quad \frac{n + 9}{2n} = \frac{-5}{2n}
\]

This gives us the equation \( n + 9 = 2n - 5 \). Add 5 and subtract \( n \) from both sides to get \( n = 14 \) ←answer.

2. A number divided by 4 is 12 less than the number.

→ Solution: Let the number be \( n \). We translate the given statement in Math notations as before:

\[
\text{A number divided by 4 is 12 less than the number} \quad \frac{n}{4} = n - 12
\]

Therefore,

\[
\frac{n}{4} = n - 12 \quad \text{or} \quad n = 4(n - 12)
\]

\[
n = 4n - 48 \quad \text{Hence} \; 3n = 48 \quad \text{or} \quad n = 16.
\]

3. Sum of three consecutive integer is 324. Find the numbers.

→ Solution: Suppose the smallest integer is \( x \). Then next two consecutive integers are \( x + 1 \) and \( x + 2 \). Sum = \( x + (x + 1) + (x + 2) = 3x + 3 \). We are given that the sum is 324. Therefore,

\[
3x + 3 = 324 \quad \Rightarrow \quad 3x = 321 \quad \text{or} \quad x = \frac{321}{3} = 107
\]

→ The integers are 107, 108 and 109 ans.

4. A house cleaning service charges a base fee of $50 plus $60 per hour to clean a standard single family house. If the total bill for cleaning Mike’s house from this company was $300 before taxes, how many hours did it take to clean the house?

→ Solution: In this problem, the unknown quantity is the number of hours it took to clean the house. Let us select \( h \) to represent the number of hours. Then we construct an equation using the given information that will allow us to solve for \( h \). Study the way solution is written:

Let \( n \) be the number of hours needed for the cleaning.

\[
\text{Base charge + hourly charge} = \text{total bill}
\]

\[
50h + 60h = 290 \quad \Rightarrow \quad 60h = 290 - 50 = 290
\]

\[
\quad \Rightarrow \quad h = 4.
\]

5. Helena wants to fence a rectangular field for exercising her dogs that are boarded overnight. She has 130 ft of fencing to use for the perimeter of the region. What should the dimensions of the region be if she wants the length to be 15 ft greater than the width?

→ Solution: Let \( W \) be the width of the region. Then the length \( L \) is 15 ft more than \( w \), or \( L = W + 15 \).

Since the perimeter \( P = 130 \) ft,

\[
130 = 2(L + W) \quad \text{or} \quad 130 = 2((W + 15) + W)
\]

Therefore, \( 130 = 2(W + 15 + W) \) or \( 130 = 2(2W + 15) \)

Which gives, \( 130 = 4W + 30 \) or \( 100 = 4W \)

This shows that \( W = 25 \) and \( L = W + 15 = 25 + 15 = 40 \).
1.4.3 Homework

1. Eight less than 3 times a number is 6 times the number, increased by 10. Find the number.

2. A number increased by 6 is 3 less than twice the number. Find the number.

3. A number divided by 3 is 6 less than the number. 27. Find the number.

4. Students who attend academic conferences receive $150 for lodging plus $0.50 per mile driven. How many miles did Jose drove if he was reimbursed $255 for an overnight trip?

5. Adam receives a weekly salary of $400 at Anderson’s Appliances. He also receives a 6% commission on the total dollar amount of all sales he makes. What must his total sales be in a week if he is to make a total of $790?
1.5 Variation

1.5.1 Tutorial

Ratio of \(a\) to \(b\) is a fraction \(\frac{a}{b}\). Two variables are in direct variation (or direct proportion) if their ratio remains constant. Symbolically we write \(x\) varies directly to \(y\) as \(x :: y\). In that case, \(y = kx\) for some constant \(k\). \(k\) is called the constant of variation.

For example, let \(x\) denote the number of T-shirts and \(y\) denote the cost in US dollars. Then \(x\) and \(y\) are in direct variation. Suppose for \(x = 30\), \(y = \$210\), then the constant of variation is the cost of one T-shirt which is \(\$7\). We write \(y = 7x\) US dollars.

Variables \(x, y\) are in inverse variation (or proportion) if \(y = \frac{k}{x}\) or \(xy = k\). In that case we write \(x \propto y\).

Number of people \(x\) and time \(t\) to complete a task are in inverse proportion.

Suppose \(P = \) number of people and \(D = \) number of days they take to paint a four bedroom colonial house.

If it is given that 5 people can paint a four bedroom colonial house in 8 days, then \(k = PD = (5)(8) = 40\). Thus we have a formula \(PD = 40\). If there were 10 people working on the house, then they will take 4 days.

\[
\begin{array}{|c|c|}
\hline
\text{direct variation} & x :: y & y = kx \\
\text{Inverse variation} & x \propto y & xy = k \\
\hline
\end{array}
\]

Table 3: Variation formulas

**Variation story:** In a school mountain climbing picnic, Tia offered her service to carry food bags, her friends thought she was crazy because food bags were real heavy. As days passed, students consumed food and Tia had less to carry. Rest of the students had to carry same weight throughout the picnic. Tia knew that the weight of the food bags and time varied inversely!
### 1.5.2 Examples

1. Suppose $P$ and $Q$ varies directly. When $P = 35$, $Q = 7$. Find the value of $Q$ when $P = 91$.

→ **Solution:** Direct variation gives us the relation $P = k \cdot Q$. Now when $P = 35$, $Q = 7$. Therefore,

$$35 = k(7) \text{ or } k = \frac{35}{7} = 5.$$ 

Hence we have $P = 5Q$. When $P = 91$,

$$91 = 5Q \implies Q = \frac{91}{5} = 18.2 \text{ ans.}$$

2. $C$ varies inversely as $J$. If $C = 7$ when $J = 0.7$, determine $C$ when $J = 12$.

→ **Solution:** Inverse variation gives us the relation $C = \frac{k}{J}$. Now when $C = 7$, $J = 0.7$. Therefore,

$$7 = \frac{k}{0.7} \text{ or } k = 7 \times 0.7 = 4.9.$$ 

Thus we have $C = \frac{4.9}{J}$. Now when $J = 12$,

$$C = \frac{4.9}{12} = 0.4 \text{ ans.}$$

3. $y$ varies directly as the square of $R$. If $y = 4$ when $R = 4$, determine $y$ when $R = 8$.

→ **Solution:** Direct variation is between $y$ and $R^2$, so the relation is $y = k \cdot R^2$. Now when $y = 4$, $R = 4$. Therefore,

$$4 = k(4^2) \text{ or } k = \frac{4}{16} = 0.25.$$ 

So we have $y = 0.25R^2$. When $R = 8$,

$$y = 0.25(8^2) \implies y = 0.25(64) = 16 \text{ ans.}$$

4. Suppose $T$ varies inversely as the square root of $S$. If $T = 5$ when $S = 4$, determine $T$ when $S = 16$.

→ **Solution:** Inverse variation is between $T$ and $\sqrt{S}$, so the relation is $T = \frac{k}{\sqrt{S}}$. Now when $T = 5$, $S = 4$. Therefore,

$$5 = \frac{k}{\sqrt{4}} \text{ or } 5 = \frac{k}{2} \text{ which implies } k = 10.$$ 

Hence, we have $T = \frac{10}{\sqrt{S}}$. Now when $S = 16$,

$$T = \frac{10}{\sqrt{16}} = \frac{10}{4} = 2.5 \text{ ans.}$$

5. Suppose $y$ varies directly as the square root of $t$ and inversely as $s$. If $y = 12$ when $t = 36$ and $s = 2$, determine $y$ when $t = 81$ and $s = 4$.

→ **Solution:** This is a joint variation problem. Direct variation is between $y$ and $\sqrt{t}$ and inverse variation between $y$ and $s$. Therefore the relation is, $y = \frac{k\sqrt{t}}{s}$.

Now when $y = 12$, $t = 36$ and $s = 2$. Therefore,

$$12 = \frac{k\sqrt{36}}{2} \text{ or } 12 = \frac{6k}{2} \text{ which gives us } 12 = 3k \text{ or } k = 4.$$ 

So we have $y = \frac{4\sqrt{t}}{s}$. Now when $t = 81$ and $s = 4$,

$$y = \frac{4\sqrt{81}}{4} = 9 \text{ ans.}$$
1.5.3 Homework

1. Suppose $A$ and $B$ varies directly. When $A = 20$, $B = 5$. Find the value of $B$ when $A = 36$.

2. Suppose $x$ varies inversely as $y$. If $x = 4$ when $y = 2.5$, determine $x$ when $y = 2$.

3. $U$ varies directly as the square of $V$. If $U = 8$ when $V = 4$, determine $U$ when $V = 6$.

4. Suppose $C$ varies inversely as the square root of $D$. If $C = 2$ when $D = 36$, determine $C$ when $D = 25$.

5. Suppose $y$ varies directly as the square root of $t$ and inversely as $s$. If $y = 8$ when $t = 49$ and $s = 14$, determine $y$ when $t = 125$ and $s = 10$. 
1.6 Chapter Test

1. Simplify:
   (a) $4(6 - 3 + 2) - 3(8 + 2 - 11)$.
   (b) $2 - \frac{5 + 8}{3} + 7$.

2. Solve:
   (a) $5y - 4 = 14 - y$
   (b) $3(x - 3) + 4 = 2(x + 4)$
   (c) $4(x - 3) - 2(x - 1) = 8$
   (d) $4(x - 4) + 12 = 4(x - 1)$
   (e) $5(t + 2) - 14 = 2t - 1$
   (f) $\frac{x + 3}{3} = \frac{x + 2}{4}$.
   (g) $\frac{x - 4}{x + 3} = \frac{2}{3}$.
   (h) $3x + 2 - 6x = -x - 15 + 8 - 5x$
   (i) $6x + 8 - 22x = 28 + 14x - 10 + 12x$

3. Suppose the final balance formula is $A = P(1 + rt)$. Determine $P$ when $A = \$3,600$, $r = 0.04$, and $t = 5$.

4. Volume of a circular cylinder is given by the formula $V = \pi r^2 h$. Find the value of $h$ when $V = 942$, $\pi = 3.14$, and $r = 5$.

5. Temperature conversion formula is given as $F = \frac{1}{1.8}C + 32$, where $F$ stands for Fahrenheit and $C$ for Celsius. Determine $F$ when $C = 7$. Determine $C$ when $F = 77$.

6. Slope of a line passing through $(x_1, y_1)$ and $(x_2, y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$. Determine $m$ when $y_2 = 8$, $y_1 = -4$, $x_2 = -3$, and $x_1 = -5$.

7. Ronnie McNeil pays 8 cents to make a copy of a page at a copy shop. She is considering purchasing a photocopy machine that is on sale for $250, including tax. How many copies would Ronnie have to make in the copy shop for her cost to equal the purchase price of the photocopy machine she is considering buying?

8. Marty received an inheritance of $12,000. If he wants to invest three times as much money in stocks as in mutual funds, how much money should Marty invest in mutual funds and in stocks?

9. Each year, Andrea donates a total of $1000 for scholarships at Mercer County Community College. This year, she wants the amount she donates for scholarships for liberal arts to be three times the amount she donates for scholarships for business. Determine the amount she will donate for each type of scholarship.


11. $A$ varies jointly as $R_1$ and $R_2$ and inversely as the square of $L$. Determine $A$ when $R_1 = 120$, $R_2 = 8$, $L = 5$, and $k = 3.2$

12. $t$ varies directly as the square of $d$ and inversely as $f$. If $t = 192$ when $d = 8$ and $f = 4$, determine $t$ when $d = 10$ and $f = 6$.

13. $y$ varies directly as the square root of $t$ and inversely as $s$. If $y = 12$ when $t = 36$ and $s = 2$, determine $y$ when $t = 81$ and $s = 4$.


15. $y$ varies directly as the square of $R$. If $y = 4$ when $R = 4$, determine $y$ when $R = 8$.

16. $H$ varies directly as $L$. If $H = 12$ when $L = 40$, determine $H$ when $L = 10$. 
2 Problem Solving

2.1 Four Step Method

2.1.1 Tutorial

George Polya (1887-1985) developed a four step method for solving any problems.

Step 1: Understand the problem
- Re-write the problem in a language that is clear and makes sense to you.
- Identify what you need to find and what you know.
- Organize all information provided and/or draw a picture or diagram.

Step 2: Devise a plan
- Think about a similar problem you may have solved in past and apply that method.
- Look at all the information given to brainstorm an idea to solve the problem.
- or Assign variables to unknown quantities and design a step by step plan.

Step 3: Carry out the plan
- Use algebra and solve an equation or use a formula.
- Look for a pattern and plug in some values.
- Try trial and error to estimate and adjust a solution.
- or Use a diagram, table, or apply some other concept.

Step 4: Look back
- Does your answer completely answer the problem.
- and Does the answer make sense.

A puzzle: The following puzzle had me confused for a long time, then I realized that there was an error in calculation!!

On father’s day, three siblings went to a store to buy a clock for their father. The price tag was $30, so each chipped in $10 from their pocket, paid $30 and walked out. But the clock was $5 off, so the store manager sent a salesman after them to give back $5. The salesman thought that since $5 was not divisible by 3, he gave back $3 and kept $2. This way each sibling paid $9 and the salesman kept $2, but $9 × 3 = 27, plus 2 equals 29. What happened to the 30th dollar?
2.1.2 Examples

1. One side of a rectangular playground is 10 yards longer than the other side. If the perimeter is 120 yards, what are the dimensions of the playground?

→ Solution: Following the Four Step Method

Step 1 Understand the Problem:
We know that a rectangle has four sides and opposite sides are the same, so the dimensions of a rectangle are the length and width. We know that the perimeter of a rectangle is the measurement of the outer boundary, i.e. the sum of all four sides. Therefore we need to find the length and width so that the sum of all four sides is 120 yards.

Step 2 Devise a Plan:
We can use algebra and geometry to solve this. Suppose the width is \( x \) yards long. Then the length is \( x + 10 \) yards. In the following figure, we represent this information:

![Figure 1: Rectangular playground](image)

\[(x + 10) + (x) + (x + 10) + (x) = 120 \quad (1)\]

Step 3 Carry Out the Plan:
We solve equation given using our knowledge of linear equations:

\[
(x + 10) + (x) + (x + 10) + (x) = 120
\Rightarrow
4x + 20 = 120
\Rightarrow
4x = 100
\Rightarrow
x = 25
\]

So the width of the rectangle is 25 yards and the length is \( 25 + 10 = 35 \) yards.

Step 4 Look Back:
To check the answer we add all four sides and verify that the perimeter is 120 yards.

\[
25 \text{ yards} + 35 \text{ yards} + 25 \text{ yards} + 35 \text{ yards} = 120 \text{ yards}
\]

2. The length of a rectangular walled yard is 3 feet less than twice the width. The perimeter of the walled yard is 78 feet. Find the dimensions and the area of the yard.

→ Solution: Suppose the width is \( w \) feet, then the length is \( 2w - 3 \) feet. In the following figure, we represent this information:

![Figure 2: Rectangular Wall](image)

The perimeter or sum of all the four sides is 78 feet. Therefore,
\[(2w - 3) + (w) + (2w - 3) + (w) = 78.\] Hence, \[6w - 6 = 78\]
Adding 6 on either sides yields \[6w = 84\] or \[w = 14.\]

So the width is 14 feet and the length is \(2(14) - 3 = 25\) feet.

But we also need the **Area** of the rectangle = \(\text{length} \times \text{width} = 14 \times 25 = 350\) square feet.

To check the answer we add all four sides and verify that the perimeter is 78 feet.

\[14\text{ feet} + 25\text{ feet} + 14\text{ feet} + 25\text{ feet} = 78\text{ feet}.\]
2.1.3 Homework

1. The perimeter of a rectangular room is 270 yards. The longer side of the room is 15 yards longer than the shorter side. Find the dimensions of the room.

   (a) Identify the information using variables and draw a diagram.

   (b) Write the appropriate equation.

   (c) Solve the equation.

   (d) Check the solution.

2. A rectangular yard is fenced with 120 feet of barbed wire. If the length of the yard is twice as long as the width, what is the length?
3. A rectangular football field is 20 feet longer than twice the width. The perimeter of the field is 160 feet. Find the dimensions of the field.

4. A rectangular garden is surrounded by 30 feet of fencing. If length of the garden is five less than three times the width, what are the dimensions and the area of the garden?

5. The length of a rectangular dance floor is 22 feet more than the width and the perimeter is 156 feet. If carpeting will cost $99 plus $12 for each square foot, calculate the cost of carpeting the floor.
2.2 Pattern and Puzzles

2.2.1 Tutorial

Often finding a pattern is an effective problem solving strategy. Many number games are based on this strategy. We will study three areas of applications, sequence of numbers, magic squares and Sudoku puzzles.

A **sequence** is a list of numbers (called terms) which follow a specific pattern or a rule. Once the pattern is recognized, we can guess other terms. For example in the sequence, 1, 2, 1, 4, 1, 6, ..., every other number is 1 and the alternate values are increasing even numbers.

**Magic square** is an arrangement of integers written in a square formation so that the sum of each row, each column and each diagonal are equal, and each number is used only once. Here is an example of a 3 by 3 magic square which uses numbers 1 through 9:

\[
\begin{array}{ccc}
2 & 7 & 6 \\
9 & 5 & 1 \\
4 & 3 & 8 \\
\end{array}
\]

**Sudoku** is a number puzzle made up of a grid in which each row, each column and each subgrid uses every number once. A standard Sudoku puzzle is a 9 by 9 grid and uses the numbers 1 through 9, but in our course, we will only consider 4 by 4 and 6 by 6 puzzles. Here is an example of a completed 4 by 4 Sudoku:

\[
\begin{array}{cccc}
2 & 4 & 1 & 3 \\
1 & 3 & 4 & 2 \\
3 & 1 & 2 & 4 \\
4 & 2 & 3 & 1 \\
\end{array}
\]

A variant of Sudoku is called **KenKen** and we will consider 3 by 3 and 4 by 4 puzzles. A square is divided into 9 or 16 smaller squares (which are called cells). The solver enters 1, 2, 3 or 1, 2, 3, 4 in these cells so that each row and each column contains 1, 2, 3 (or 1, 2, 3, 4). Some cells are surrounded by thick lines which forms a partition. If you see ‘1−’ in one cell of any partition, then the difference of entries in those cells should be 1. But if you see ‘6×’ in one cell of any partition, then the product of entries in those cells should be 6. Here is a 4 by 4 KenKen puzzle.

\[
\begin{array}{cccc}
1− & 7+ & 4× \\
2÷ &  \\
3− & 2÷ & 8+ \\
\end{array}
\]

Research shows that keeping the brain active by doing puzzles slow downs the progression of Alzheimer’s disease or dementia in old age, so cultivate a habit of solving Crossword, Sudoku or other logic puzzles. If you have any senior citizens in your family, visit them and do puzzles with them. Puzzles like these can be found everywhere in newspapers, magazines or books in the dollar store, or on the web.

For the more details on Kenken puzzles, visit [http://www.kenkenpuzzle.com](http://www.kenkenpuzzle.com)
2.2.2 Examples

1. Find the next two numbers of the following sequence and explain the pattern:

1, 2, 4, 8, _______, _______

→ Solution: We can see that each number is multiplied by 2 to get the next number. We show our work by making arrows as shown below:

\[
\begin{array}{cccccc}
1 & \times 2 & 2 & \times 2 & 4 & \times 2 \\
 & & 4 & \times 2 & 8 & \times 2 \\
 & & & & 16 & \times 2 \\
 & & & & & 32
\end{array}
\]

So the next two numbers will be 16 and 32.

2. Complete the following magic square using numbers 1 through 9:

\[
\begin{array}{ccc}
8 & 5 & 2 \\
6 & & \\
& & 9
\end{array}
\]

→ Solution: The sum of the entries in all the three rows = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45. Since these rows have equal sum value, this value is \(\frac{45}{3} = 15\).

Now suppose the second entry in the last row is \(x\). Then \(2 + x + 6 = 15\). Therefore, \(x = 7\). But the column sum is also 15. Therefore, the second entry in the last column is \(15 - (8 + 6) = 1\). Now for the diagonal, to find the first entry in the first row, which is \(15 - (5 + 6) = 4\). After filling in those entries, you can complete the puzzle as shown below:

\[
\begin{array}{ccc}
4 & 3 & 8 \\
9 & 5 & 1 \\
2 & 7 & 6
\end{array}
\]

3. Solve the Sudoku puzzle given using the numbers 1 through 4.

\[
\begin{array}{ccc}
 & 2 & \\
3 & & \\
4 & & \\
\end{array}
\]

→ Solution: Because there is already a 4 in the first column, in the top left grid, the 4 must be above the 3. Since there is a 2 in the first row, in the top left grid, the 2 must be left of the 3. That means 1 must be in the corner of the top left grid. Then the remaining open squares in both the first row and the first column must be 3. Continuing this process results in the solution below.
4. Solve the KenKen puzzle given in the tutorial 2.2.1.

→ **Solution:** Observe that the second partition in the first column with condition 3—has 2 entries. We must enter two numbers from \( \{1, 2, 3, 4\} \) whose difference is 3. This can be done in only one way, \( 4 - 1 = 3 \). Therefore, we can enter 1 and 4 in these two cells. But we don’t know the order in which these numbers can be entered, so we wait till we get more information. Now the first column has 1, 2, 3, 4 and 4, 1 are in the last two cells, so first two cell contains 2 and 3. Now the third partition in the first row with condition 4× has three entries and product of these entries is 4. This can be done in two ways, \( 4 \times 1 \times 1 \) or \( 1 \times 2 \times 2 \). In both cases, we have repetition of one entry, which must be written in different rows and different columns. But 2 is in the first or second row, so the third entry in row one is 1, fourth entry is 4 and the second entry in the last column is 1. Similar logical analysis and elimination process gives us the following solution:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 4 & 2 \\
4 & 1 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 4 & 1 \\
2 & 4 & 2 \\
3 & 2 & 3 \\
\end{array}
\]
2.2.3 Homework

1. Find the next two numbers and explain the pattern:
   (a) 7, 4, 1, -2, _____, _____

   (b) 1, 4, 9, 16, _____, _____

   (c) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{8}$, _____, _____

   (d) 0, 1, 8, 27, _____, _____

2. Use the integers from 1 to 9 to complete the following magic squares.
   (a)
   
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

   (b)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

3. Use the even integers from 2 to 18 to complete the magic square

<table>
<thead>
<tr>
<th>4</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6</td>
</tr>
</tbody>
</table>

4. Use the integers from 1 to 4 to complete the Sudoku.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>
5. Use 1, 2 and 3 to solve the following KenKen:

6. Use 1, 2, 3 and 4 to solve the following KenKen:
2.3 Venn Diagram

2.3.1 Tutorial

In statistical inquiries, we often see statements connected by ‘AND’, ‘OR’ or ‘NOT’. For example

1. 20 students drink coffee and juice for breakfast
2. 30 students drink coffee or tea for breakfast
3. 10 students drink tea but not juice for breakfast

Such information describes different groups (called sets). We can effectively present these groups and their relationships using Venn diagrams. A Venn diagram represents information pictorially in circles, one for each group, within an enclosing rectangle. Common elements of the groups are represented by the areas of overlap among the circles.

Here is an example of a Venn diagram representing three groups, students who drink (C) coffee, (T) tea or (J) fruit juice in their breakfast. The overlapping area which is common to two circles or to three represents those who belong to two groups or to all three groups.
2.3.2 Examples

1. In a class of 70 students, 45 are enrolled in Mat 111, 55 are enrolled in Phy 120 and 36 students are enrolled in both Mat 111 and Phy 120. Present this information using a Venn diagram and answer the following questions:

(a) How many are enrolled in Mat 111 but not in Phy 120?
(b) How many are enrolled in Phy 120 but not in Mat 111?
(c) How many are not enrolled in either Mat 111 or Phy 120?

→ Solution: There are two groups of students, one group are those enrolled in Mat 111 and the second are those enrolled in Phy 120. So we need two circles in our Venn diagram, one for M (taking Math), and one for P (taking Physics) all enclosed by a rectangle (C) which represents the whole class of 70 students.

Circle $M$ stands for students enrolled in Mat 111 and circle $P$ stands for students enrolled in Phy 120. Therefore, there are 45 elements in $M$, 55 elements in $P$ and 36 in the common area. We enter 36 in the overlap area of the two circles as shown here:

Since there are 45 total in $M$ and 36 in the overlap, then to the left of the overlap area in $M$ we write $45 - 36 = 9$. Since there are 55 total in $P$ then on the right of the overlap area in $P$ we write $55 - 36 = 19$. Adding these three together we get $9 + 36 + 19 = 64$. But since there needs to be a total of 70 in the square, we write $70 - 64 = 6$ outside of these two circles in $C$. The completed Venn diagram is shown below and we can use this to answer the questions.
Now we answer questions (a), (b) and (c):

(a) How many are enrolled in Mat 111 but not in Phy 120? Ans. 9.
(b) How many are enrolled in Phy 120 but not in Mat 111? Ans. 19.
(c) How many are not enrolled in either Mat 111 or Phy 120? Ans. 6.
2.3.3 Homework

1. Survey of 100 individuals reveal that 58 like Coke, 50 like Pepsi and 18 like both. Complete the Venn diagram and answer the following questions:

(a) How many like Coke but not Pepsi?
(b) How many like Pepsi but not Coke?
(c) How many do not like either Coke or Pepsi?

2. A survey asked 170 people what movies they liked. 78 liked action (A); 96 liked horror (H); 67 liked comedy (C); 30 liked comedy & horror; 45 liked action & comedy; 42 liked horror & action; 29 liked all three. Complete the following Venn diagram and answer how many:

(a) Didn’t like any of these?
(b) Liked action & comedy but not horror?

3. Out of 150 Lincoln students, 90 have a laptop, 70 have a tablet and 30 have both- a laptop and a tablet. Draw a Venn diagram and answer the following questions:

(a) How many have a laptop but not tablet?
(b) How many have a tablet but not laptop?
(c) How many do not have either a laptop or a tablet?
4. A survey asked 150 people what movies they liked. 80 liked action (A); 60 liked horror (H); 75 liked comedy (C); 30 liked comedy & horror; 30 liked action & comedy; 25 liked horror & action; 10 liked all three. Draw a Venn diagram then answer how many:

   (a) Didn’t like any of these?
   (b) Liked action & comedy but not horror?

5. A survey asked 150 people what candidate they liked. 80 liked candidate Smith; 60 liked candidate Jones; 75 liked candidate Martin; 30 liked Jones & Martin; 30 liked Smith & Martin, 25 liked Smith & Jones; 10 liked all three. Draw a Venn diagram then answer how many:

   (a) Didn’t like any of these?
   (b) Liked Smith & Martin but not Jones?

6. A survey asked 155 people where they went on vacation in the last year. 107 the beach; 90 a trip; 76 visiting family; 57 beach & a trip; 54 beach & family; 52 trip & family; 35 beach, trip, & family. Draw a Venn diagram to answer how many:

   (a) went to exactly one of these
   (b) went to none of these
   (c) went on a trip or to visit family but not to the beach.
2.4 Percents

2.4.1 Tutorial

Percent (%) literally means per=divided by (÷) cent=100. Therefore, 5 percent means 5 divided by 100 which is $\frac{5}{100} = 0.05$. As ‘per’ stands for ‘divided by’, ‘of’ stands for ‘times’. So

1. To find p% of a number

\[ \text{Number} = \left( \frac{p}{100} \right) \times R, \text{ Where } R \text{ is the original or reference value} \]

For example:

5% of $750 is \((5 \div 100) \times $750 = 0.05 \times $750 = $37.50\).

2. Absolute Change in a quantity is measured by subtracting original value or reference value (R) from the current value (C). Thus

absolute change = current value-reference value.

3. Relative change is the absolute change divided by the reference value and is generally written as a percentage.

relative change = \left( \frac{\text{Absolute change}}{\text{reference value}} \right) \times 100\%

For example, if a quantity changes from 2 to 2.5,

Absolute change = 2.5 - 2 = 0.5 and Relative Change = \((0.5 \div 2) \times 100\% = 25\%\)

Discounts and sales taxes are rate of changes. Discount is a negative relative change and sales tax is positive relative change.
2.4.2 Example

1. What is 2.8% of 70 feet?

→ Solution: Using equation 1, \( p = 2.8\% \) and \( R = 70 \), thus

\[
N = (2.8 \div 100) \times 70 \text{ ft} = 1.96 \text{ ft}.
\]

2. What percent of $250 is $190?

→ Solution: As before, using equation 1 the reference or original value \( R = 250 \), the value \( N = 190 \), and we need to find \( p \). Thus

\[
\begin{align*}
$190 &= (p \div 100) \times $250 \\
$190 &= \frac{p \times $250}{100} \\
$190 &= p \times $2.5 \\
p &= \frac{190}{2.5} = 76\% 
\end{align*}
\]

3. Find 12% of $375.

→ Solution: \( p = 12\% \) and \( R = 375 \), thus

\[
N = (12 \div 100) \times $375 = $45.
\]

4. Find 0.75% of $500.

→ Solution: \( p = 0.75\% \) and \( R = 500 \), thus

\[
N = (0.75 \div 100) \times $500 = $3.75.
\]

Note: A similar question was asked in fall 2016 final and most got it wrong mainly because students were confused with decimal point! Instead of using 0.75% = 0.75 \div 100 = 0.0075, they used 0.75. This is a common mistake, so be aware of it.

5. What percent of $300 is $37.50?

→ Solution: the reference or original value \( R = 300 \), the value \( N = 37.50 \), thus

\[
\begin{align*}
$37.50 &= (p \div 100) \times $300 \\
$37.50 &= \frac{p \times $300}{100} \\
$37.50 &= p \times $3 \\
p &= \frac{37.50}{3} = 12.5\%
\end{align*}
\]

6. A table cloth worth $12.99 is sold for $10. Find the absolute and relative change.

→ Solution: Using Equations 2 and 3, The current value is the final price, which is \( C = $10 \). The original price is the reference value, so \( R = $12.99 \). Therefore,

\[
\begin{align*}
\text{Absolute change} &= $10 - $12.99 = -$2.99 \\
\text{Relative change} &= \frac{-$2.99}{$12.99} \times 100\% = -23\%
\end{align*}
\]

We can say that discount is 23%. 

2.4.3 Homework

1. What is 3.5% of 80 feet?

2. What percent of $450 is $75?

3. Find 15% of $575.

4. Find 0.05% of $2,500.

5. What percent of $250 is $17.50?

6. 12.5% of what amount is $15?

7. A laptop worth $1,250 is sold for $1,050. Find the absolute and relative change.
2.5 Chapter Test

1. Find the next two numbers and explain the pattern:

   (a) 5, 8, 11, 14, ______, ______.

   (b) 16, 10, 4, -2, ______, ______.

   (c) 1, 3, 9, 27, ______, ______.

2. Survey of 90 individuals revealed that 45 use laptop \((L)\), 50 use tablet \((T)\) and 10 use both. Complete the Venn diagram and answer the following questions:

   \[
   \begin{array}{c}
   T \\
   L
   \end{array}
   \]

   (a) How many use laptop but not tablet?

   (b) How many do not use laptop?

   (c) How many do not use either laptop or tablet?

3. Joan paid $342 for a laptop which were listed for $299. Find the local tax percent she paid.
4. A survey asked 120 people what movies they liked. 65 liked action (A); 55 liked horror (H); 56 liked comedy (C); 24 liked comedy & horror; 31 liked action & comedy; 27 liked horror & action; 16 liked all three. Draw a Venn diagram then answer how many:

(a) Didn’t like any of these?

(b) Liked action & comedy but not horror?

![Venn Diagram]

5. Use the integers from 1 to 9 to complete the magic square

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

6. Perimeter of a rectangular room is 92 yards. Longer side of the room is 14 yards longer than the shorter side. Find the dimensions of the room. Also find the area of the room.

7. Answer the following questions:

(a) What is 22% of $400?

(b) $68 is what percent of $420?

(c) $42.5 is 17% of what amount?

8. A laptop which was originally priced for $650 was 14.5% off. Find its sale price.
3 Consumer Mathematics

3.1 Simple Interest

3.1.1 Tutorial

When money is invested, investor earns interest. There are various types of interest, the most basic model is **simple interest**. In this model, the interest is calculated on the basis of the invested amount called principal.

If $P_0$ is the initial deposit or principal $R\%$ is the annual percentage to be given as an interest, then every year the interest amount will be $P_0 \times r$. After $t$ number of years, the interest will be $I_t = P_0 \times r t$ and the final balance will be $P_t = P_0 + I_t$.

**Formulas:**

<table>
<thead>
<tr>
<th>Simple Interest $I_t = P_0 \times r t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final balance $P_t = P_0 (1 + rt)$</td>
</tr>
</tbody>
</table>
3.1.2 Examples

1. Julie invested $450 in a simple interest plan for 12 years at 6% APR. Find the interest and the final balance.

   → Solution: Here, \( P_0 = 450 \), APR = 6% (hence \( r = 0.06 \)) and \( t = 12 \) years. Therefore,
   \[
   I_{12} = P_0rt = (450)(0.06)(12) = 324.
   \]
   Final balance \( P_{12} = P_0 + I_{12} = 450 + 324 = 774 \).

2. Haida borrowed $800 from Mike with 8% APR of a simple interest. How much will she pay back after 3 months?

   → Solution: Three months is \( \frac{3}{12} = 0.25 \) years. So the interest will be calculated with \( P_0 = 800 \), \( r = 0.06 \) and \( t = 0.25 \).
   \[
   I_{0.25} = P_0rt = (800)(0.06)(0.25) = 12 \text{ and the final balance } = 812. 
   \]
   She will pay back 812 dollars.

3. Johnny invested $ 500 in a simple interest plan and received a final balance of $ 900 after 10 years. Find the rate of interest, APR.

   → Solution: This time, \( P_0 = 500 \), \( P_{10} = 900 \) and \( t = 10 \) years. So, \( I_{10} = 900 - 500 = 400 \). Therefore,
   \[
   I_{10} = P_0rt \implies 400 = r(500)(10) \implies 400 = 5000r \implies r = \frac{400}{5000} = 0.08 \text{ or APR } = 8\%.
   \]
   The annual percentage rate is 8% ← ans.

4. How long will it take to get $500 interest on an investment of $ 4,000 at 5% APR in a simple interest account?

   → Solution: We substitute \( I_t = 500 \), \( P_0 = 4000 \) and \( r = 0.05 \) in our formula.
   \[
   I_t = P_0rt \implies 500 = 4000(0.05)t \implies 200t \implies t = \frac{500}{200} = 2.5
   \]
   It will take 2 and half years.

5. Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

   → Solution: First, it is important to know that interest rates are usually given as an annual percentage rate (APR), the total interest that will be paid in the year. Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.
   \[
   P_0 = 1000 \text{ (the principal)}
   r = 0.02 \text{ (2% rate per half-year)}
   t = 8 \text{ (4 years = 8 half-years)}
   \]
   Therefore, \( I_8 = 1000(0.02)(8) = 160 \). You will earn $160 interest total over the four years.
3.1.3 Homework

1. Jane invested $500 for 10 years in a simple interest account offering 5% APR. Find the interest and her final balance.

2. John invested $3,500 for 8 years in a simple interest account at $2\frac{1}{2}$% APR. How much interest will he earn? Find his final balance.

3. After 12 years, Hardy received an interest of $1,470 from his investment of $2,000 in a simple interest account. Fine his APR.

4. Kate invested $800 in a simple interest account with 6% APR. After how many years she will get a final balance of $3000?
5. Find an APR which turns a principal of $12,000 invested in a simple interest account in 10 years to a final balance of $21,370.

6. Paris wants to earn a simple interest \( I = $3,100 \) on a principal of $30,000. If the APR is 8%, how many years should she invest to reach her goal?

7. On a deposit of $3,500 for 8 years, John is offered a simple interest of 2%. How much interest will he earn? Find his final balance.

8. John borrowed $760 from a bank with 4% APR. He paid off his loan after 120 days. Calculate his total due.
3.2 Compound Interest

3.2.1 Tutorial

In a simple interest investment, we do not get (or pay) interest on interest earned on our investment. In a compound interest, we add interest to the previous balance. If interest is added $N$ times in a year, we say that the interest is compounded $N$ times. Thus compounded quarterly means compounded four times and $N = 4$. Here is the formula for computing compound interest.

\[
\text{Compound Interest } P_N = P_0 \left(1 + \frac{r}{k}\right)^{Nk}
\]

Value of $k$ depends on the phrase “compounded —” in the statement of a given problem. We list some values of $k$:

<table>
<thead>
<tr>
<th>Phrase</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compounded annually</td>
<td>1</td>
</tr>
<tr>
<td>Compounded half-yearly</td>
<td>2</td>
</tr>
<tr>
<td>Compounded quarterly</td>
<td>4</td>
</tr>
<tr>
<td>Compounded monthly</td>
<td>12</td>
</tr>
<tr>
<td>Compounded daily</td>
<td>365</td>
</tr>
</tbody>
</table>
3.2.2 Examples

1. Macbeth invested $1,320 at 6% APR compounded quarterly for 8 years. Find the interest and the final balance.

→ Solution: Observe that \( P_0 = $1,320, \ r = 6\% = 0.06, \ K = 4 \) (there are 4 quarters in a year) and \( N = 8 \) years. Now,

\[
P_8 = P_0 \left(1 + \frac{r}{K}\right)^{Nk} \implies P_8 = 1,320 \left(1 + \frac{0.06}{4}\right)^{(8)(4)}
\]

\[
\implies P_8 = 1,320(1.015)^{32} = 1,320(1.61032432) = $2125.63.
\]

Therefore, interest \( I = $2125.63 - $1,320 = $805.63. \)

2. Joan deposits $7,500 in a monthly compounding investment with 6% APR. In how many years the amount will be doubled?

→ Solution: Trial and error method

In this example, \( P_0 = $7,500,\ P_N = $15,000 \) and \( r = 0.056. \)

\[
P_N = P_0 \left(1 + \frac{r}{K}\right)^{Nk} \implies 15000 = 7,500 \left(1 + \frac{0.06}{12}\right)^{12N}
\]

Divide both sides by 7,500 to get 2 = (1.005)^{12N}

We try different values of \( N \) (use Excel).

\( (1.005)^{120} = 1.8 \ N = 10 \)

\( (1.005)^{132} = 1.97 \ N = 11 \) which is close.

\[\implies N = 11 \text{ years.}\]

Second solution

\[
P_N = P_0 \left(1 + \frac{r}{K}\right)^{Nk} \implies 15000 = 7,500 \left(1 + \frac{0.06}{12}\right)^{12N} \implies 2 = (1.005)^{12N}
\]

Now use logarithmic function ln (or LN) on both sides to get,

\[
0.69314718 = (0.004987542)(12N) \text{ or } N = 11.58
\]

3. Marc Invested $125,000 for 12 years at 5% compounded quarterly. Find the accumulated balance and interest.

→ Solution:

\[
P_{12} = P_0 \left(1 + \frac{r}{K}\right)^{Nk} \implies P_{12} = 125,000 \left(1 + \frac{0.05}{4}\right)^{48} = 125,000(1.0125)^{48} = 125,000(1.8153548530)
\]

\[
= $226,919.36
\]

Interest \( I = P_{12} - P_0 = $226,919.36 - $125,000 = $101,919.36 \)

4. How much investment (deposit) guarantees a final balance of $10,000 in ten years, if the APR is 4% compounded quarterly?

→ Solution: Notice that we need to find \( P_0. \) \( P_{10} = P_0 \left(1 + \frac{r}{K}\right)^{Nk}. \) Substitution gives,

\[
10,000 = P_0(1 + \frac{0.04}{4})^{40} = P_0(1.488863734)
\]

\[\therefore P_0 = \frac{10,000}{1.488863734} = 6716.531387\]

It requires $6716.53 to get $10,000 in 10 years.
3.2.3 Homework

1. Jose invested $1,500 for 5 years in a quarterly compounding account offering 6% APR. How much interest he will earn? Find his final balance.

2. On a deposit of $5,000 for 8 years, Joan is offered a monthly compounding of $2\frac{1}{2}$%. How much interest she will earn? Find her final balance.

3. Walter invested $6,000 in an annual compound interest account for 12 years. The APR offered to him is 7.5%. Find his gain (interest) and determine how much he will get at the end.

4. In 10 years, Joseph received a final balance of $10,353.25 as a result of his investment of $2000 in a daily compounding account. Find his APR.
5. Annie invested $2,500 in an annual compound interest account which gave her 6% APR. In how many years can she earn an interest of $1,000?

6. Find an APR which turns a principal of $12,000 invested in a daily compound interest account in 10 years to a final balance of $21,000.

7. Hagar wants to earn $150,000 from an investment of $30,000. Government bonds offer 6% APR with a monthly compounding. In how many years will he reach his goal?

8. Jack and Jill invested $5,000 each for 10 years. Jack invested in a simple interest account with 8% APR and Jill invested in a daily compound interest account with 4% APR. Calculate their final balances and determine who earned more.
3.3 Annuity and long term saving plans

3.3.1 Tutorial

Investments in which we save a fixed amount every month for number of years are called **annuity** or **Savings plans**. Sometimes annuity refers to the the final balance or a contract from any financial agency.

There are three formulas in this section:

<table>
<thead>
<tr>
<th>Accumulated balance</th>
<th>$P_N = \frac{d[(1+\frac{r}{k})^{Nk} - 1]}{\left(\frac{r}{k}\right)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly deposit</td>
<td>$d = \frac{A_N\left(\frac{r}{k}\right)}{(1+\frac{r}{k})^{kN} - 1}$</td>
</tr>
<tr>
<td>Payoff Annuity</td>
<td>$P_0 = \frac{d[1 - (1+\frac{r}{k})^{-Nk}]}{\left(\frac{r}{k}\right)}$</td>
</tr>
</tbody>
</table>

Individual retirement accounts (which are also called IRA or 401(K) accounts) are the most popular savings accounts. We open a savings account with a financial firm of our employees’s choice. Every month, a certain portion of our income is directly deposited to our account. There is no income tax charged to the investment, until you make withdrawal. Part of your retirement income comes from such investment. The retirement age is 65 years. If a 20 years old individual has an IRA, the first formula computes the total savings after 45 years. If he/she wants to have certain amount for 20 years after his retirement, the third formula determines the amount he/she should have at the time of retirement.
3.3.2 Examples

1. Hans saved $50 every month for 15 years earning 5.7% annual percentage rate (APR). Find his final balance.

→ **Solution:** Observe that regular deposit \( d = 50 \), APR=5.7% (so \( r = 0.057 \)) and \( N = 15 \). Substituting these values in the savings plan formula, we get,

\[
P_N = \frac{d[(1 + \frac{r}{k})^{Nk} - 1]}{(\frac{r}{k})} = \frac{50[(1 + \frac{0.057}{12})^{(15)(12)} - 1]}{(0.00475)} \approx \frac{50(1.13001) - 1}{0.00475} = 50(1.346619501 - 1)(0.00475) = 24,701.26 \text{ Answer}
\]

2. Mr. Least Bothered decided to accumulate $120,000 in 20 years for his daughter’s college tuition. One investment firm offered him 4.8% APR. Help him find his monthly deposit (\( d \)).

→ **Solution:** Observe that \( P_{20} = 120,000 \), APR=4.8%, \( r = 0.048 \), \( k = 12 \) and \( N = 20 \). Substituting these values in the savings plan formula for \( d \), we get,

\[
d = \frac{P_{20} (\frac{r}{k})}{[(1 + \frac{r}{k})^{Nk} - 1]} = \frac{120,000 (\frac{0.048}{12})}{[(1 + \frac{0.048}{12})^{(12)(20)} - 1]} = \frac{120,000(0.004)}{[(1.004)^{240} - 1]} = \frac{480}{1.606700133} = 298.75 \text{ Answer}
\]

3. Andre wants $1,000 per month for 30 years after he is retired. His IRA offers him 3.6% APR. At the time of retirement, how much savings Andre should have in his account?

→ **Solution:** Observe that \( d = 1,000 \), APR=3.6%, \( r = 0.036 \), \( k = 12 \) and \( N = 30 \). Substituting these values in the payoff annuity formula for \( P_0 \), we get,

\[
P_0 = \frac{d[1 - (1 + \frac{r}{k})^{-Nk}]}{(\frac{r}{k})} = \frac{1,000[1 - (1 + \frac{0.036}{12})^{-360}]}{(0.036)^{12}} = \frac{1,000[1 - 0.3401450]}{0.003} = \frac{1,000[0.6598550]}{0.003} = 219,951.66 \text{ Answer}
\]

4. Jacob wants to save $200,000 by saving $200 a month in an investment earning 6% APR. How long will it take?

→ **Solution by trial and error method.** Observe that \( P_N = 200,000 \), \( d = 200 \) and \( r = 6\% = 0.06 \). Substituting these values (with \( N \) as it is) in the savings plan formula (provided), we get,

\[
200,000 = \frac{200[(1 + \frac{0.06}{12})^{12N} - 1]}{(\frac{0.06}{12})} = \frac{200(1.005)^{(12N)} - 1}{0.005}
\]

Multiplication by .005 gives 1000 = 200(1.005)^{(12N)} - 1

\[\Rightarrow 1001 = 200(1.005)^{(12N)}, \text{ Or } 5.005 = (1.005)^{(12N)}\]

For \( N = 25 \), \( (1.005)^{12N} = 4.4649698121623049771420869730681 \)
For \( N = 27 \), \( (1.005)^{12N} = 5.0327343742406936784902574549 \)

We see that \( N=27 \) years provides a value that is very close to 5.005 compared to \( N=25 \). Hence \( N=27 \) years is a correct guess.
3.3.3 Homework

1. John saved $30 a month for 40 years at 4% APR in a savings account. Find his accumulated balance and calculate the total interest he received.

2. Ann accumulated balance of $100,000 in 35 years in a savings plan which offered an APR of 3.6%. Find her monthly deposit \( d \).

3. Nasir wants $1,200 per month retirement income for 25 years after he is retired. His IRA offers 4.8% APR. At the time of retirement, how much savings Nasir should have in his account?
3.4 Loan Installment

3.4.1 Tutorial

Loan payment $d$ (see page 212):

$$d = \frac{P_0 \left(\frac{r}{k}\right)}{\left[1 - (1 + \frac{r}{k})^{-kN}\right]}$$

In most cases, we will assume $k = 12$.

Most common loans are (1) student loans (2) auto finance and (3) mortgage loans. Students loans are generally for 3 to 4 years. Their APR is lowest among all kind of loans. Auto loans are also for 4 to 5 years, but the interest rates are very high. Mortgage loans are for 15, 20 or 30 years. Interest rates can be reduces by making a down payment. Good credit history also helps.
3.4.2 Examples

1. Lisa financed a car worth $30,000 for 4 years with 6.6% APR. Find her monthly payment.

→ Solution: Here the loan amount $P_0 = $30,000, APR=6.6%, $r=0.066$ and $N=4$. Substituting these values in the loan formula (provided), we get,

$$d = \frac{P_0 \left( \frac{r}{12} \right)}{\left[ 1 - \left(1 + \frac{r}{12}\right)^{12N} \right]} = \frac{30,000 \left( \frac{0.066}{12} \right)}{\left[ 1 - \left(1 + \frac{0.066}{12}\right)^{48} \right]}$$

$$= \frac{30,000(0.0055)}{\left[ 1 - (1.0055)^{-48} \right]} = \frac{30,000(0.0055)}{1 - 0.76852953}$$

$$= \frac{165}{0.231476746} = \$712.83 \ \text{Answer}$$
3.4.3 Homework

1. Jade mortgage her house for $200,000 for 30 years with an APR of 4.8%. Find her monthly installment.

2. Walter financed his car worth $30,000 for 5 years with an APR of 6%. Find his monthly installment.
3.5 Chapter Test

1. Henry deposits $1,300 at 1.5% APR in a simple interest account for 4 years. Find the interest \( I \) and the final balance \( P_4 \).

2. With an initial investment of $2,500, how long will it take to accumulate $5,000 in an account which is paying simple interest of 8%?

3. After a discount of 15%, I paid $510 for a TV. Find the original price of the TV.

4. John invested $12,000 for 8 years at 3.6% compounded quarterly. Find the final balance.

5. What should the initial investment be to accumulate $8,000 in 10 years in an account with 5.4% APR compounded monthly?

6. Tania saved $70 per month in a savings plan giving 4% APR for 25 years. Find her accumulated balance and total interest she earned.

7. Rudy is planning for his retirement. He would like to get $1,500 per month for 25 years after he retires. What must be his total savings if a 401(K) plan offers him 3% APR?

8. Jerry financed his car worth $40,000 for four years with 3.6% APR. Find his monthly installment. Also compute the finance charges.
4 Descriptive Mathematics

4.1 Describing Data Values

4.1.1 Tutorial

By describing we mean finding important properties of data that can help us in decision making. For example, if we know average weight of a Dell desktop, we can decide how many a forklift can carry. Graphs (such as bar chart, pie chart) of profit of last 10 years can help us with a quick comparison between two companies. We will study three methods to describe data:

- Graphical representation of data.
  - Frequency distribution.
  - Bar graphs, pie chart and histograms.
- Measure of central tendency of data.
  - Mean, mode, median and quartiles.
- Variation (spread or fluctuation) in data.
  - Range and standard deviation.

**Frequency Table:**
A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

**Bar graph:**
A bar graph is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

**Pie Chart:**
A pie chart is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.
4.1.2 Examples

1. Construct a frequency table and plot a bar diagram and pie chart for hourly wages of 20 employees. 10, 9, 11, 10, 9, 10, 11, 10, 10, 9, 11, 8, 12, 10, 11, 8, 10, 9, 12. Also find the mode of the distribution.

→ **Solution:** Frequency of a data value is how often it appears. For example 8 appears 2 times, so the frequency of 8 is 2 (this was done by observation).

<table>
<thead>
<tr>
<th>Wages (in $)</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employees</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Mode of the distribution is a value or values that appears most often. In this example hourly wage 10 appears 7 times which is more often than any other values. Therefore, 10 is the mode. To plot a bar diagram, we will use x-axis for data values (hourly wages) and y-axis for the frequencies.

![Bar Diagram](image)

*Figure 3: Hourly wages (in $)*

We will now draw a pie chart for the same data. First we find relative frequencies of each data values.

<table>
<thead>
<tr>
<th>Wages (in $)</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employees</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>5</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>Rel. freq.</td>
<td>10%</td>
<td>20%</td>
<td>35%</td>
<td>25%</td>
<td>10%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Relative frequency of value:

\[
\text{Relative Frequency} = \frac{\text{Frequency of } x}{\text{Total Frequency}}
\]
For example relative frequency of \( \frac{7}{20} = 0.35 = 35\% \). The pie chart is given in the figure 2.

![Pie chart](image)

**Figure 4: Pie chart**

In a separate worksheet we will learn how to make bar and pie chart using MS Excel. We will also learn to make histograms and line graphs using the software.
4.1.3 Homework

Construct frequency distribution and draw a bar graph and pie chart for each of the following data set:

(a) Hourly wages: 12, 10, 8, 11, 10, 11, 10, 9, 10, 8, 9, 11, 10, 11, 8, 9, 10, 10, 8, 11.

(b) Annual income (in thousand dollars): 45, 47, 48, 46, 48, 47, 46, 45, 49, 48, 50, 45, 46, 47, 46, 47, 50, 46, 48, 47, 45, 48, 49, 46, 50, 45, 48.
(c) Working hours: 7, 5, 10, 8, 9, 8, 6, 7, 9, 9, 6, 9, 8, 10, 5, 7, 5, 8, 9, 10, 6, 9, 8, 7.
4.2 Central Values

4.2.1 Tutorial

To describe data by the central theme is very natural to us. We often say that the average speed is 60 miles per hour or most prefer caffeine free drink. In this worksheet we will learn about average or mean.

Mean

Mean or average is a measure of central tendency of a distribution. Mean of a population is denoted by $\mu$ and that of a sample is denoted by $\bar{x}$. When data values are given in row form,

$$\mu \text{ or } \bar{x} = \frac{\text{sum of } x\text{-values}}{\text{number of } x\text{-values}} = \frac{\sum x}{n},$$

where $n$ stands for the number of values and $\sum$ is a sigma symbol which means sum.

In the following example 3, we noticed that we can multiply frequencies and corresponding data values to find the total of values in a discrete distribution. This will modify our previous formula for the mean as

$$\bar{x} = \frac{\sum(x \cdot f)}{N}$$

where, $\sum(x \cdot f)$ is a sum of numbers in the $x \times f$-column.
4.2.2 Examples

1. Find $\bar{x}$ for the distribution 4, 6, 3, 4, 6, 7, 4, 3, 2, 3.

→ Solution: Sum of $x$ values = 4+6+3+4+6+7+4+3+2+3 = 42.
There are 10 values. Therefore,

$$\bar{x} = \frac{42}{10} = 4.2. \leftarrow \text{ans.}$$

2. The average weight of a class of 30 students is 145 lb. Five new students are added to the class whose weights are 151 lb, 152 lb, 154 lb, 148 lb and 145 lb. Find the new average weight of the class.

→ Solution: By definition, the average weight of 30 students is:

$$145 = \frac{\text{total weight}}{30}$$

So the total weight of 30 students is $145 \times 30 = 4350$. Weight added to the class is $151 + 152 + 154 + 148 + 145 = 750$ lb. This implies that the total weight of all the 35 students is $4350 + 750 = 5100$ lb. Therefore, the new average weight of 35 students is:

$$\frac{5100}{35} = 146.71 \text{ lb.} \leftarrow \text{ans.}$$

3. Find the $\bar{x}$ for the following distribution:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

→ Solution: If we write the data values in a row form, we will get the following distribution: 0, 0, 1, 1, 2, 2, 3, 3, 3, 4. Therefore,

$$\bar{x} = \frac{0 + 0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 + 3 + 4}{12}$$

$$= \frac{2(0) + 2(1) + 3(2) + 4(3) + 1(4)}{12}$$

$$= \frac{0 + 2 + 6 + 12 + 4}{12} = \frac{24}{12} = 2 \leftarrow \text{ans.}$$

4. Mart’s average score for the College basketball season 1 is 32 points and for season 2 is 28 points. In season 1, she played 14 games and in season 2 she played 8 games. What is her combined average score?

→ Solution: Total points for the season 1 are $32 \times 14 = 448$ and for the season 2 are $28 \times 8 = 224$. Therefore, she scored $448 + 224 = 672$ points in 22 games. This gives us an average of

$$\mu = \frac{672}{22} = 30.5 \leftarrow \text{ans.}$$
5. Find the mean of

<table>
<thead>
<tr>
<th>No. of children</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of families</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

→ **Solution** We complete the table by adding a column of \( x \times f \).

\[
\begin{array}{|c|c|c|}
\hline
x & f & x \times f \\
\hline
0 & 4 & 0 \\
1 & 7 & 7 \\
2 & 9 & 18 \\
3 & 5 & 15 \\
4 & 3 & 12 \\
5 & 2 & 10 \\
\hline
N = 30 & \sum x f = 62 \\
\hline
\end{array}
\]

Therefore,

\[
\bar{x} = \frac{62}{30} = 2.07 \leftarrow \text{ans.}
\]

**Note:** Microsoft Excel has a built-in function which can find mean. We have solved the above example using excel:

6. Use excel to find the mean of

<table>
<thead>
<tr>
<th>No. of children</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of families</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

→ **Solution:** Follow the following steps:

(a) Enter the data in a single column of an excel work sheet starting from the second cell A2.

(b) Last entry will be in A31 cell.

(c) In cell A32, write

\[
= \text{AVERAGE}(A2:A31)
\]

(d) Click on AVERAGE. End you will get answer ‘2.066666667’ which is what we got before.
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No. of children</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>14</td>
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<td>15</td>
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<td>30</td>
<td>5</td>
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<tr>
<td>31</td>
<td>5</td>
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<td></td>
</tr>
<tr>
<td>32</td>
<td>2.066666667</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.2.3 Homework

1. Find the mean in each cases:
   (a) 4, 6, 8, 10, 8.
   (b) 3, 5, 6, 8, 7, 9, 6, 3, 5, 6, 8, 7, 9, 6, 3, 5, 6, 8, 7, 9, 6, 3, 5, 6, 8, 7, 9.

(c) | Hourly wages | 9 | 10 | 11 | 12 | 13 | 14 |
    | No. of employees | 2 | 5 | 6 | 4 | 2 | 1 |

2. Average weight of 25 students in Mat 106 class was 140 lb. New five students joined the class. Their weights are 148 lb, 154 lb, 160 lb, 147 lb and 153 lb. Find the new average weight of the class.
4.3 Median And Quartiles

4.3.1 Tutorial

**Five number summary:** When data values are arranged in ascending order, we can describe a data set by five values located in five key positions. These positions are

- First position, minimum value
- Second positioned at the end of the first quarter called the first quartile $Q_1$.
- Third positioned in the middle, called the median $M$.
- Fourth positioned at the end of the third quarter called the third quartile $Q_3$.
- Fifth position, the maximum value.

Line on either sides are called whiskers. If both whiskers are of equal length, then the data values form a normal distribution. If left whisker is longer, then the distribution is skewed on left (or negatively skewed). If right whisker is longer, then the distribution is skewed on right (or positively skewed). Figure 5 is a box plot of a negatively skewed data.

**Median** is a middle value of a data distribution. If there are two middle values, then the average of these two values is the median. To find median, first arrange the values in an ascending order, then

- Find the **median position**. If given values are $x_1, x_2, \ldots x_N$, then,
  
  \[ \text{the median position} = l = \frac{N + 1}{2}. \]

- If $l$ is an integer, then $x_l$ is the median, otherwise, median is an average of $x_t$ and $x_{t+1}$ where $t$ is the largest integer smaller than $l$.

**Quartiles** are positioned half way on either sides of the median. If we arrange data in a line (as shown in the following figure) $M =$ median lies in the middle. $Q_1$ in the middle of the first half and $Q_3$ in the middle of the second half. Therefore $Q_1 =$ median of the first half and $Q_3 =$ median of the other half.

![Figure 5: Positions of five points](image)

To find $Q_1$ and $Q_3$, follow the following procedure:

- Find the median $M$.
- Find the median of all the data values which are smaller than $M$. This value is $Q_1$.
- Find the median of all the data values which are larger than $M$. This value is $Q_3$. 
4.3.2 Examples

1. Find $M$, $Q_1$ and $Q_3$ of: $3, 6, 6, 4, 5, 7, 8$.

→ **Solution:** To find the median we need to arrange the data values in the ascending order:

$$3, 4, 5, 6, 6, 7, 8$$

$N = 7$, so the median position is $\frac{N+1}{2} = \frac{8}{2} = 4$.
Therefore, median is the $4^{th}$ data value which is 6.

Now, $Q_1$ is the median of $3, 4, 5$ which is 4 and $Q_3$ is the median of $6, 7, 8$ which is 7.
Therefore, $M = 6, Q_1 = 4$ and $Q_3 = 7$. The box plot is given as follows:

![Box plot for the data $3, 6, 6, 4, 5, 7, 8$.](image)

2. Draw the box plot for the data $10, 9.6, 8.2, 12, 11, 10.2$.

→ **Solution:** Values in the ascending order are

$$8.2, 9.6, 10, 10.2, 11, 12.$$ 

There are six values. Therefore the median position is $\frac{N+1}{2} = \frac{7}{2} = 3.5$ Therefore the median is equal to

$$\frac{x_3 + x_4}{2} = \frac{10 + 10.2}{2} = 10.1.$$ 

Now, $Q_1$ is the median of $8.2, 9.6, 10$ which is 9.6 and $Q_3$ is the median of $10.2, 11, 12$ which is 11.
Therefore, $M = 10.1, Q_1 = 9.6$ and $Q_3 = 11$.

![Box plot for the data $10, 9.6, 8.2, 12, 11, 10.2$.](image)
3. Find the five number summary of the following distribution and draw the box plot.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

→ **Solution:** As in the case of mean, we can write the data values in a row as shown below, and find the median.

1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 4, 4, 5.

Since there are 15 values, the median position is \( \frac{15+1}{2} = 8 \).

Therefore, the median is 8th value which is 2.

Now, \( Q_1 \) is the median of 1, 1, 1, 1, 2, 2, 2 which is 1 and \( Q_3 \) is the median of 2, 3, 3, 3, 4, 4, 5 which is 3.

Therefore, \( Min = 1, Max = 5, M = 2, Q_1 = 1 \) and \( Q_3 = 3 \).

![Figure 8: Positions of five numbers](image)
4.3.3 Homework

1. Draw the box plot for the following data. Is the distribution skewed? On which side?
   (a) 4, 5, 2, 2, 4, 5, 3, 5, 6, 4.

   (b) 7.5, 5.5, 7.2, 2.8, 4.5, 3.7, 4.5, 8.6, 9.3, 6.6, 3.4.

   (c) 4, 3, 2, 2, 5, 3, 5, 6, 9, 7, 5, 1, 2, 4, 3, 5, 6, 10, 8, 5, 7, 5, 4, 5, 9, 8, 6, 4.
<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>2</td>
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<td>7</td>
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<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
<th>4.5</th>
<th>5.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
4.4 Standard Deviation

4.4.1 Tutorial

Standard deviation measures how spread the data values are. For example the information “average score of a class of 32 students is 67 points with a standard deviation of 2 points” can be interpreted as “at least 24 (75% of 32) students must have scored between 63(average - 2(standard deviation)) and 71 (average + 2(standard deviation))”.

The simplest measure of variation is a range of values.

**Example:** Three patients in a hospital has following pulse rates: Patient-1: 70, 85, 100, Patient-2: 78, 85, 92 and Patient-3: 82, 85, 88. Find range of the pulse rate of these patients.

Notice that all three patients are having same mean $\bar{x} = 85$ pulses per minutes and same median pulse rate (also 85). Do you think that they are in same condition? Why patient 3 is in better condition than patient 1? We can differentiate three patients using range of their pulse rate.

→ **Solution:**

1. Patient-1: Range=100-70=30.

Range is easy and a quick way to find the spread of the values. It ignores mean position. For example, range can not say whether more values are below average ($\bar{x}$) or above the average.

**Example:** A paint manufacturer claims that the average drying time for his paint is less than 2 hrs. A sample range is 2.5 hrs. This does not give us any indication if most values are below 2 hrs. or above 2 hrs. It is important to know if most values are close to $\bar{x}$.

Therefore, it is useful to design a unit that measures how far data values are from the mean value. This parameter must have

- large value if the fluctuations are high.
- small value if the data values are cluster around the mean value.

This can be achieved by considering average distance from the mean value. But the sum of all such distances is always zero. That is,

$$\sum (x - \bar{x}) = 0.$$

Therefore, the average distance is also zero!
Example: Verify the above equality for 2, 4, 4, 6, 7, 7.

→Solution:

\[ \bar{x} = \frac{2 + 4 + 4 + 6 + 7 + 7}{6} = \frac{30}{6} = 5. \]

Therefore,

\[ \sum (x - \bar{x}) = (2 - 5) + (4 - 5) + (4 - 5) + (6 - 5) + (7 - 5) + (7 - 5) \]
\[ = -3 - 1 - 1 + 1 + 2 + 2 = 0. \]

We define the **population standard deviation** \( \mu \) and the **sample standard deviation** \( s \) to be

\[ \sigma = \sqrt{\frac{\sum (x - \mu)^2}{N}} \quad \text{and} \quad s = \sqrt{\frac{\sum (x - \bar{x})^2}{N - 1}} \]

**Standard Deviation: Step by Step**

- Calculate the mean, \( \bar{x} \) (or \( \mu \)).
- Write a table that subtracts the mean from each observed value.
- Square each of the differences.
- Add this column.
- Divide by \( N - 1 \) (or \( N \)) where \( N \) is the number of items in the sample (or population).
- Take the square root.
4.4.2 Examples

1. Following are four determinations of the specific gravity of aluminum 2.64, 2.70, 2.67 and 2.63. Find
   (a) the range.
   (b) Population standard deviation.

→ Solution:
   (a) Range = 2.70 - 2.63 = 0.07.

(b) Since there are only four values, it must be a sample. So we will calculate sample standard deviation.

First we find mean
\[ \bar{x} = \frac{2.64 + 2.70 + 2.67 + 2.63}{4} = 2.66 \]

In the following table, we find \((x - \bar{x})^2\).

<table>
<thead>
<tr>
<th></th>
<th>2.64</th>
<th>2.70</th>
<th>2.67</th>
<th>2.63</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x - \bar{x}</td>
<td>-0.02</td>
<td>0.04</td>
<td>0.01</td>
<td>-0.03</td>
<td></td>
</tr>
<tr>
<td>(x - \bar{x})^2</td>
<td>0.0004</td>
<td>0.0016</td>
<td>0.0001</td>
<td>0.0009</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Therefore, \(\sigma = \sqrt{\frac{0.003}{4}} = 0.027 \leftarrow \text{ans.}\)

2. The owner of the Hot & Spice restaurant wants to find out how much people spend at the restaurant. He
   examines 10 randomly selected billing receipts and writes down the following data.
   \$44, \$50, \$38, \$96, \$42, \$47, \$40, \$39, \$46, \$50.

→ Solution: He calculated the mean by adding and dividing by 10 to get
\[ \bar{x} = \$49.20 \]

Below is the table to find the standard deviation:

<table>
<thead>
<tr>
<th></th>
<th>(x - 49.2)</th>
<th>((x - 49.2)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>44</td>
<td>-5.2</td>
<td>27.04</td>
</tr>
<tr>
<td>50</td>
<td>0.8</td>
<td>0.64</td>
</tr>
<tr>
<td>38</td>
<td>11.2</td>
<td>125.44</td>
</tr>
<tr>
<td>96</td>
<td>46.8</td>
<td>2190.24</td>
</tr>
<tr>
<td>42</td>
<td>-7.2</td>
<td>51.84</td>
</tr>
<tr>
<td>47</td>
<td>-2.2</td>
<td>4.84</td>
</tr>
<tr>
<td>40</td>
<td>-9.2</td>
<td>84.64</td>
</tr>
<tr>
<td>39</td>
<td>-10.2</td>
<td>104.04</td>
</tr>
<tr>
<td>46</td>
<td>-3.2</td>
<td>10.24</td>
</tr>
<tr>
<td>50</td>
<td>0.8</td>
<td>0.64</td>
</tr>
</tbody>
</table>

\[ \sum(x - \bar{x})^2 = 2600.4 \]
Therefore,

\[
\text{Standard deviation } = s = \sqrt{\frac{\sum(x - \bar{x})^2}{N - 1}} = \sqrt{\frac{2600.4}{9}} = \sqrt{288.7} = \$16.99
\]

Statistically, this will be interpreted as follows:
Most customers probably spend between
\[
\bar{x} - s = \$49.20 - \$16.99 = \$32.21
\]
and
\[
\bar{x} + s = \$49.20 + \$16.99 = \$66.19
\]
4.4.3 Homework

Find the population and sample standard deviation of the following data:

(a) $4, 5, 2, 3, 3, 5, 6, 4$.

(b) $1, 7, 4, 4, 6, 5, 8$.
4.5 Chapter Test

1. Annual starting salaries (in thousands of dollars) for social workers with a bachelors degree and no experience are shown below:
   28, 29, 30, 32, 34, 29, 30, 32, 34, 29, 31, 33, 34, 29, 30, 31, 33, 35, 29, 30, 31, 33, 35

   (a) Construct a frequency distribution and find the mode.

   (b) Draw a bar diagram of the frequency distribution.

2. Waiting time of 11 customers at a customer service station are 2, 6, 3, 5, 5, 1, 8, 6, 10, 7, 9 minutes. Find five point summary and draw a box plot of the data. Is the distribution skewed? On which side?.
3. Monthly income (in thousand dollars) of 9 employees are given below.
2, 2, 5, 5, 6, 8, 8, 9, 9.

(a) Find the average income.

(b) Find the value of sample standard deviation (use $s = \sqrt{\frac{\sum (x - \text{mean})^2}{n-1}}$).

(c) Find the value of population standard deviation (use $\sigma = \sqrt{\frac{\sum (x - \text{mean})^2}{n}}$).
4. Following pie chart represents the daily wages of 200 employees of a packaging warehouse:

![Pie Chart](image-url)

(a) How many employees have hourly wage of $9?

(b) Find the mean amount of the distribution.

5. A mean average of 60 on five exams is needed to pass a course. On her first four exams, Tina received grades of 51, 57, 72, and 65.

(a) What grade must she receive on her last exam to pass the course?
(b) An average of 80 is needed to get a B in the course. Is it possible for Tina to get a B? If so, what grade must she receive on the fifth exam?

(c) If her lowest grade of the exams already taken is to be dropped, what grade must she receive on her last exam to pass the course?

(d) If her lowest grade of the exams already taken is to be dropped, what grade must she receive on her last exam to get a B in the course?
5 Probability

5.1 Basic concept

5.1.1 Tutorial

**Statistical inquiry** is a study or an experiment with well defined (but random) outcomes. For example, if we are measuring length of Tsetse flies, we can be sure that the outcome (length) will be a positive number. When we actually start gathering measurements, it is impossible to predict the length of a next specimen. This is why we say that the measurements are random numbers. From our experience we know that it is not likely that this number would ever be larger than half inch or the probability of finding a fly longer than half inch is zero.

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

**Events and Outcomes**

- The result of an experiment is called an **outcome**.
- An **event** is any particular outcome or group of outcomes.
- A **simple event** is an event that cannot be broken down further.
- The **sample space** is the set of all possible simple events.

A die (plural dice) has 6 sides with marks 1 through 6 on it. A single die can roll 1, 2, 3, 4, 5 or 6.

![Picture of a die](image)

Figure 9: Picture of a die (dice is the plural of die)

**Study the following example:**

If we roll a standard 6-sided die, describe the sample space and some events.

**Solution:**

- The sample space is the set of all possible simple events: \{1,2,3,4,5,6\}
• simple events:
  We roll a 1: \{1\}
  We roll a 5: \{5\}

• compound events:
  We roll a number bigger than 4: \{5,6\}
  We roll an even number: \{2,4,6\}

**Basic Probability**

Given that all outcomes are equally likely, we can compute the probability of an event $E$ using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally likely outcomes}}$$

**Certain and Impossible events**

- An impossible event has a probability of 0.
- A certain event has a probability of 1.
- The probability of any event must be $0 \leq P(E) \leq 1$
5.1.2 Examples

1. If we roll a 6-sided die, calculate
   (a) $P(\text{rolling a } 1)$
   (b) $P(\text{rolling a number bigger than } 4)$

→ Solution: Recall that the sample space is $\{1,2,3,4,5,6\}$
   (a) There is one outcome corresponding to “rolling a 1” = \{1\}, so the probability is \[ \frac{1}{6} \]
   (b) There are two outcomes bigger than a 4 = \{5,6\}, so the probability is \[ \frac{2}{6} = \frac{1}{3} \]

2. Let’s say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

→ Solution: There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is \[ \frac{14}{20} = \frac{7}{10} \]

3. A coin is tossed twice. Describe the sample space and find the probabilities of getting (a) two tails (b) Head first time and then tail (c) only one tail (d) at least one tail.

→ Solution: The sample space contains all the possible results. Denote $H$ for head and $T$ for tail. Then the sample space is \{HH, HT, TH, TT\}.
   (a) Outcome of getting a tail both times is denoted by $TT$ and the probability of getting two tails is $P(TT) = \frac{1}{4}$.
   (b) First Head and then Tail can be denoted by $HT$ which is one outcome out of 4. Therefore, the probability of getting $HT$ is $P(HT) = \frac{1}{4}$.
   (c) One $T$ can be obtained in two ways, $HT$ and $TH$. So the probability of getting one tail is $P(\text{one } T) = \frac{2}{4} = \frac{1}{2}$.
   (d) At least one $T$ can be obtained in three ways, namely $HT$, $TH$ and $TT$. Therefore, the probability of getting at least one tail is $P(\text{at least one } T) = \frac{2}{4} = \frac{1}{2}$. 
4. A coin is tossed twice. What is the probability of getting one $T$?

→ **Solution:** Here event $E =$ one tail $= \{HT \text{ or } TH\}$. Both $P(HT)$ and $P(TH)$ are equal to $\frac{1}{4}$. Therefore,

$$P(\text{one } T) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2} \rightarrow \text{Answer.}$$

**Cards:**
A standard deck of 52 playing cards consists of four suits (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different rank: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

5. Compute the probability of randomly drawing one card from a deck and getting an Ace.

→ **Solution:** There are 52 cards in the deck and 4 Aces so

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769.$$  

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.
5.1.3 Homework

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.

   (a) A red ball is drawn

   (b) A white ball is drawn

2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:

   (a) A consonant

   (b) A vowel

3. A group of people were asked if they had run a red light in the last year. 150 responded “yes”, and 185 responded “no”. Find the probability that if a person is chosen at random, they have run a red light in the last year.

4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dots, and 40 indicated they don’t enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.

5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.
6. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>8</td>
<td>18</td>
<td>13</td>
<td>39</td>
</tr>
<tr>
<td>Female</td>
<td>10</td>
<td>4</td>
<td>12</td>
<td>26</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
<td>22</td>
<td>25</td>
<td>65</td>
</tr>
</tbody>
</table>

7. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

<table>
<thead>
<tr>
<th></th>
<th>Zero</th>
<th>One</th>
<th>Two or more</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>9</td>
<td>5</td>
<td>19</td>
<td>33</td>
</tr>
<tr>
<td>Female</td>
<td>18</td>
<td>10</td>
<td>20</td>
<td>48</td>
</tr>
<tr>
<td>Total</td>
<td>27</td>
<td>15</td>
<td>39</td>
<td>81</td>
</tr>
</tbody>
</table>

8. Compute the probability of tossing a six-sided die and getting an even number.

9. Compute the probability of tossing a six-sided die and getting a number less than 3.

10. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?

11. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?
5.2 Working with Events

5.2.1 Tutorial

Now let us examine the probability that an event does not happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is \( P(\text{six}) = \frac{1}{6} \). Now consider the probability that we do not roll a six: there are 5 outcomes that are not a six, so the answer is \( P(\text{not a six}) = \frac{5}{6} \). Notice that

\[
P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1
\]

Complement of an Event

• The complement of an event is the event “\( E \) doesn’t happen”.

• The notation \( \overline{E} \) is used for the complement of event \( E \).

• We can compute the probability of the complement using \( P(\overline{E}) = 1 - P(E) \).

• Notice also that \( P(E) = 1 - P(\overline{E}) \).

Probability of two independent events

Events \( A \) and \( B \) are independent events if the probability of Event \( B \) occurring is the same whether or not Event \( A \) occurs.

• \( P(A \text{ and } B) \) for independent events:

If events \( A \) and \( B \) are independent, then the probability of both \( A \) and \( B \) occurring is

\[
P(A \text{ and } B) = P(A) \cdot P(B)
\]

where \( P(A \text{ and } B) \) is the probability of events \( A \) and \( B \) both occurring, \( P(A) \) is the probability of event \( A \) occurring, and \( P(B) \) is the probability of event \( B \) occurring.

• \( P(A \text{ or } B) \):

The probability of either \( A \) or \( B \) occurring (or both) is

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
\]
Conditional Probability

- **Conditional Probability**
  The probability the event $B$ occurs, given that event $A$ has happened, is represented as $P(B | A)$.
  This is read as “the probability of $B$ given $A$”

- **Conditional Probability Formula**
  If Events $A$ and $B$ are not independent, then

  $$P(A \text{ and } B) = P(A) \cdot P(B | A)$$
5.2.2 Examples

1. If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

→ Solution: There are 13 hearts in the deck, so

\[ P(\text{heart}) = \frac{13}{52} = \frac{1}{4} \]

The probability of not drawing a heart is the complement:

\[ P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4} \]

2. Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

→ Solution:
We could list all possible outcomes: \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.
Notice there are \(2 \times 6 = 12\) total outcomes. Out of these, only 1 is the desired outcome, so the probability is

\[ \frac{1}{12} \]

3. Are these events independent?

(a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

→ Solution:
The probability that a head comes up on the second toss is \(1/2\) regardless of whether or not a head came up on the first toss, so these events are independent.

(b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

→ Solution:
These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

(c) You draw a card from a deck, then draw a second card being red without replacing the first.

→ Solution:
The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.
4. In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

→ **Solution:** The probability of choosing a white pair of socks is \( \frac{6}{10} \)

The probability of choosing a white tee shirt is \( \frac{3}{7} \)

The probability of both being white is \( \frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35} \)

5. Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin or a 6 on the die.

→ **Solution:**
There are still 12 possible outcomes: \( \{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\} \).
By simply counting, we can see that 7 of the outcomes have a head on the coin or a 6 on the die or both – we use or inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is \( \frac{7}{12} \)

As we would expect, \( \frac{1}{2} \) of these outcomes have a head, and \( \frac{1}{6} \) of these outcomes have a 6 on the die. since it contains both a head and a 6; the probability of both a head and rolling a 6 is \( \frac{1}{12} \)

\[
P(\text{head or 6}) = P(\text{head}) + P(6) - P(\text{head and 6})
\]

So

\[
P(\text{head or 6}) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12} = \frac{7}{12}
\]

6. Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

→ **Solution:**
Half the cards are red, so

\[
P(\text{Red}) = \frac{26}{52}
\]

There are four kings, so

\[
P(\text{King}) = \frac{4}{52}
\]

There are two red kings, so

\[
P(\text{Red and King}) = \frac{2}{52}
\]

We can then calculate

\[
P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King})
\]

\[
P(\text{Red or King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}
\]
7. The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

<table>
<thead>
<tr>
<th></th>
<th>Speeding ticket</th>
<th>No speeding ticket</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red car</td>
<td>15</td>
<td>135</td>
<td>150</td>
</tr>
<tr>
<td>Not red car</td>
<td>45</td>
<td>470</td>
<td>515</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>60</strong></td>
<td><strong>605</strong></td>
<td><strong>665</strong></td>
</tr>
</tbody>
</table>

(a) Has a red car and got a speeding ticket

→ **Solution:**
We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is

\[ P(\text{had a red car and got a speeding ticket}) = \frac{15}{665} \approx 0.0226 \]

(b) Has a red car or got a speeding ticket

→ **Solution:**
We could have found this probability by:

\[ P(\text{had a red car or got a speeding ticket}) = P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket}) \]

\[ P(\text{had a red car or got a speeding ticket}) = \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665} \approx 0.2932 \]

8. The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

<table>
<thead>
<tr>
<th></th>
<th>Speeding ticket</th>
<th>No speeding ticket</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red car</td>
<td>15</td>
<td>135</td>
<td>150</td>
</tr>
<tr>
<td>Not red car</td>
<td>45</td>
<td>470</td>
<td>515</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>60</strong></td>
<td><strong>605</strong></td>
<td><strong>665</strong></td>
</tr>
</tbody>
</table>

(a) Has a speeding ticket given they have a red car

→ **Solution:**
Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

\[ P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1 \]

(b) Has a red car given they have a speeding ticket

→ **Solution:**
Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

\[ P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25 \]
9. If you pull 2 cards out of a deck, what is the probability that both are spades?

→ **Solution:**
   The probability that the first card is a spade is
   \[
   \frac{13}{52}
   \]
   The probability that the second card is a spade, given the first was a spade, is
   \[
   \frac{12}{51}
   \]
   since there is one less spade in the deck, and one less total cards. The probability that both cards are spades is
   \[
   \frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588
   \]

10. A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

<table>
<thead>
<tr>
<th></th>
<th>Positive test</th>
<th>Negative test</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pregnant</td>
<td>70</td>
<td>4</td>
<td>74</td>
</tr>
<tr>
<td>Not Pregnant</td>
<td>5</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>75</td>
<td>18</td>
<td>93</td>
</tr>
</tbody>
</table>

   (a) \(P(\text{not pregnant} \mid \text{positive test result})\)

→ **Solution:**
   Since we know the test result was positive, we are limited to the 75 women in the first column, of which 5 were not pregnant.
   \[
P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} = \frac{1}{15} \approx 0.067
\]

   (b) \(P(\text{positive test result} \mid \text{not pregnant})\)

→ **Solution:**
   Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.
   \[
P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263
\]
5.2.3 Homework

1. Compute the probability of rolling a 12-sided die and getting a number other than 8.

2. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?

3. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, what is the probability that a student chosen at random did NOT earn a C?

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>8</td>
<td>18</td>
<td>13</td>
<td>39</td>
</tr>
<tr>
<td>Female</td>
<td>10</td>
<td>4</td>
<td>12</td>
<td>26</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
<td>22</td>
<td>25</td>
<td>65</td>
</tr>
</tbody>
</table>

4. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, what is the probability that a person chosen at random has at least one credit card?

<table>
<thead>
<tr>
<th></th>
<th>Zero</th>
<th>One</th>
<th>Two or more</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>9</td>
<td>5</td>
<td>19</td>
<td>33</td>
</tr>
<tr>
<td>Female</td>
<td>18</td>
<td>10</td>
<td>20</td>
<td>48</td>
</tr>
<tr>
<td>Total</td>
<td>27</td>
<td>15</td>
<td>39</td>
<td>81</td>
</tr>
</tbody>
</table>

5. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?

6. A fair coin is flipped twice. What is the probability of showing heads on both flips?
7. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?

8. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>8</td>
<td>18</td>
<td>13</td>
<td>39</td>
</tr>
<tr>
<td>Female</td>
<td>10</td>
<td>4</td>
<td>12</td>
<td>26</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
<td>22</td>
<td>25</td>
<td>65</td>
</tr>
</tbody>
</table>

9. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

<table>
<thead>
<tr>
<th></th>
<th>Zero</th>
<th>One</th>
<th>Two or more</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>9</td>
<td>5</td>
<td>19</td>
<td>33</td>
</tr>
<tr>
<td>Female</td>
<td>18</td>
<td>10</td>
<td>20</td>
<td>48</td>
</tr>
<tr>
<td>Total</td>
<td>27</td>
<td>15</td>
<td>39</td>
<td>81</td>
</tr>
</tbody>
</table>

10. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.

11. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.
12. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that a student chosen at random is female or earned a B.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>8</td>
<td>18</td>
<td>13</td>
<td>39</td>
</tr>
<tr>
<td>Female</td>
<td>10</td>
<td>4</td>
<td>12</td>
<td>26</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
<td>22</td>
<td>25</td>
<td>65</td>
</tr>
</tbody>
</table>

13. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that a person chosen at random is male or has no credit cards.

<table>
<thead>
<tr>
<th></th>
<th>Zero</th>
<th>One</th>
<th>Two or more</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>9</td>
<td>5</td>
<td>19</td>
<td>33</td>
</tr>
<tr>
<td>Female</td>
<td>18</td>
<td>10</td>
<td>20</td>
<td>48</td>
</tr>
<tr>
<td>Total</td>
<td>27</td>
<td>15</td>
<td>39</td>
<td>81</td>
</tr>
</tbody>
</table>

14. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.

15. Compute the probability of drawing a King or a heart from a deck of cards.

16. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

(a)Even-numbered given that the marble is red.

(b)Red given that the marble is even-numbered.
17. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

(a) Odd-numbered given that the marble is blue.

(b) Blue given that the marble is odd-numbered.

18. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.

19. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.

20. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

21. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.
5.3 Counting Methods

5.3.1 Tutorial

When we get to the probability situations a bit later in this chapter we will need to count some very large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don’t want to do this.

Basic Counting

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

→ Solution 1: One way to solve this problem would be to systematically list each possible meal:

<table>
<thead>
<tr>
<th>soup</th>
<th>hamburger</th>
<th>soup</th>
<th>sandwich</th>
<th>soup</th>
<th>+ quiche</th>
</tr>
</thead>
<tbody>
<tr>
<td>soup</td>
<td>+ fajita</td>
<td>soup</td>
<td>+ pizza</td>
<td>salad</td>
<td>+ hamburger</td>
</tr>
<tr>
<td>salad</td>
<td>+ sandwich</td>
<td>salad</td>
<td>+ quiche</td>
<td>salad</td>
<td>+ fajita</td>
</tr>
<tr>
<td>salad</td>
<td>+ pizza</td>
<td>breadsticks</td>
<td>+ hamburger</td>
<td>breadsticks</td>
<td>+ sandwich</td>
</tr>
<tr>
<td>breadsticks</td>
<td>+ quiche</td>
<td>breadsticks</td>
<td>+ fajita</td>
<td>breadsticks</td>
<td>+ pizza</td>
</tr>
</tbody>
</table>

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

→ Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

<table>
<thead>
<tr>
<th></th>
<th>hamburger</th>
<th>sandwich</th>
<th>quiche</th>
<th>fajita</th>
<th>pizza</th>
</tr>
</thead>
<tbody>
<tr>
<td>soup</td>
<td>soup+b</td>
<td>soup+sw</td>
<td>soup+qc</td>
<td>soup+fajita</td>
<td>soup+pizza</td>
</tr>
<tr>
<td>salad</td>
<td>salad+b</td>
<td>salad+sw</td>
<td>salad+qc</td>
<td>salad+fajita</td>
<td>salad+pizza</td>
</tr>
<tr>
<td>breadsticks</td>
<td>bs+b</td>
<td>bs+sw</td>
<td>bs+qc</td>
<td>bs+fajita</td>
<td>bs+pizza</td>
</tr>
</tbody>
</table>

if we didn’t really care what the possible meals are, only how many possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution.
Solution 3: Let’s draw a tree diagram:

This is called a “tree” diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches and then for each of those branches we drew three more branches. We count the number of branches at the final level and get 15.

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are \( m \) items in the first category and \( n \) items in the second category, then the total number of available choices is \( m \cdot n \).

Odds

1. odds in favor of an event \( E \):

\[
\frac{P(E)}{P(\bar{E})} \quad \text{or} \quad \frac{P(E)}{P(\text{not } E)}
\]

2. odds against of an event \( E \):

\[
\frac{P(\bar{E})}{P(E)} \quad \text{or} \quad \frac{P(\text{not } E)}{P(E)}
\]
3. If the odds in favor of an event $E$ are $a$ to $b$ (or $a:b$), then

(a) The odds against of the event $E$ are $b$ to $a$ (or $b:a$).

(b) The probability of event $E$ is

$$P(E) = \frac{a}{a + b}$$

.

(c) The probability of complement of event $E$ is

$$P(\bar{E}) = P(\text{not } E) = \frac{b}{a + b}$$

.
5.3.2 Examples

1. There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

→ Solution: There are 21 choices from the first category and 18 for the second, so there are

$$21 \times 18 = 378$$

possibilities.

2. Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

→ Solution: There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are

$$3 \times 5 \times 2 = 30$$

possibilities.

3. A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

→ Solution: There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in

$$2 \times 2 \times 2 = 8$$

different ways.

4. If the probability of an event E is $\frac{1}{4}$, what are the odds in favor and odds against of the event?

→ Solution:

$$P(E) = \frac{1}{4}, \quad P(\text{not } E) = 1 - P(E) = 1 - \frac{1}{4} = \frac{3}{4}$$

odds in favor:

$$\frac{P(E)}{P(\text{not } E)} = \frac{1/4}{3/4} = 1 : 3$$

odds against:

$$\frac{P(\text{not } E)}{P(E)} = \frac{3/4}{1/4} = 3 : 1$$
5. If the odds in favor of an event \( E \) is 3 to 5, what are the probability of the event \( E \) and the probability of the complement of the event \( E \)?

→ **Solution:**

\[
P(E) = \frac{3}{3 + 5} = \frac{3}{8}
\]

\[
P(\bar{E}) = P(not \ E) = \frac{5}{3 + 5} = \frac{5}{8}
\]
5.3.3 Homework

1. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?

2. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?

3. How many three-letter “words” can be made from 4 letters “FGHI” if
   (a) repetition of letters is allowed
   (b) repetition of letters is not allowed

4. How many four-letter “words” can be made from 6 letters “AEBWDP” if
   (a) repetition of letters is allowed
   (b) repetition of letters is not allowed

5. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state’s Department of Motor Vehicles?
5.4 Permutations and Combinations

5.4.1 Tutorial

Now, why would we want to use the complicated formula when it’s actually easier to use the Basic Counting Rule, as we did in the last section? Well, we won’t actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing \( r \) items out of \( n \) possibilities without replacement and where the order of selection is important or not important.

Factorial

\[
n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1
\]

Permutations

\[
nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)
\]

We say that there are \( nP_r \) permutations of size \( r \) that may be selected from among \( n \) choices without replacement when order matters.

It turns out that we can express this result more simply using factorials.

\[
nP_r = \frac{n!}{(n-r)!}
\]

Combinations

\[
nC_r = \frac{nP_r}{r!}
\]

We say that there are \( nC_r \) combinations of size \( r \) that may be selected from among \( n \) choices without replacement when order doesn’t matter.

We can also write the combinations formula in terms of factorials:

\[
nC_r = \frac{n!}{(n-r)!r!}
\]
5.4.2 Examples

1. How many different ways can the letters of the word **MATH** be rearranged to form a four-letter code word?

→ **Solution:** This problem is a bit different from previous section *counting*. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the same category (the category is: the letters of the word MATH) and each time we choose an item we do not replace it, so there is one fewer choice at the next stage:

we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are \( 4 \cdot 3 \cdot 2 \cdot 1 = 24 \) ways to spell a code word.

2. How many ways can five different door prizes be distributed among five people?

→ **Solution:** There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be

\[
5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120
\]

ways.

3. Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

→ **Solution:** Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are

\[
8 \cdot 7 \cdot 6 = 336
\]

ways the three medals can be awarded to the 8 runners.

4. I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

→ **Solution:** Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are

\[
_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024
\]

permutations.
5. How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

→ Solution: We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is

\[ 16P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680 \]

6. A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

→ Solution: Since we are choosing 4 people out of 35 without replacement where the order of selection is not important there are

\[ 35C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360 \]

combinations.

7. How many different combinations of three companies can be selected from the 16 companies?

→ Solution: We want to choose 3 companies out of 16 without replacement and where the order of selection is not important. So the answer is

\[ 16C_3 = \frac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1} = 560 \]
5.4.3 Homework

1. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?

2. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?

3. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

4. In how many ways can 4 pizza toppings be chosen from 12 available toppings?

5. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?

6. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn’t matter?

7. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn’t matter?
5.5 Chapter Test

1. If you pick one card at random from a standard deck of cards, what is the probability it will be a King of hearts?

2. If we toss a coin and roll a 6-sided die at the same time, what is the probability of getting a tail on the coin and 4 on the die?

3. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random. Find the following probabilities;

<table>
<thead>
<tr>
<th>Gender</th>
<th>zero</th>
<th>one</th>
<th>More than one</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>9</td>
<td>5</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>18</td>
<td>10</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) The person was a female.

(b) The person was male and having no credit cards.

(c) The person was male or having one credit card.

(d) Find the probability that the person was a female given that the person had more than one credit card.
4. Two 6-sided dice are rolled.
   (a) List all the possible outcomes.

   (b) Find the probability by listing all favorable outcomes, that the sum of two rolls is 6.

5. Suppose probability of an event E is $\frac{1}{4}$. What are the odds against of the event E?

6. Suppose the odds in favor of an event E are 12 to 13. What is the probability of the event E?

7. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dogs, and 40 indicated they don’t enjoy either pet. A person is chosen at random. Find the following probabilities:
   (a) The person prefer cats.

   (b) The person do not prefer dogs.
8. A bag contains 5 red balls numbered 1 to 5 and 8 blue balls numbered 1 to 8. A ball is selected at random from the bag. Find the probability the ball is

(a) odd-numbered given that the ball is blue.

(b) blue given that the ball is odd-numbered.

9. At a restaurant you can choose from 3 appetizers, 5 entrees, and 4 desserts. How many different three-course meals can you have?